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# Energy of a solid sphere under nonstationary oscillations

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Analytical solution for spherically symmetric nonstationary oscillations of acoustic and elastic solid sphere is given. Time dependence of potential, kinetic and internal energy of a solid sphere is analyzed. The received results are of practical importance for a wide range of problems connected to testing of material dynamic strength parameters and to the problems of optimizing (minimizing) energy needed for fracture of solids.

**nonstationary oscillations, analytical solution, material dynamics, dynamic elastisity, kinetic energy, potential energy, internal energy, energy exchange**

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Dynamic deformation and fracture of solids possess a number of peculiarities that make these processes significantly different from their static analogues. For example, in dynamically loaded solid media energy behaviour can be rather strange and unusual, if one keeps in mind the classic principles of mechanics [1]. As it will be demonstrated below, in the case of dynamically loaded solid sphere, internal energy can be significantly time dependent and even can periodically vanish.

This peculiarity connected with energy behaviour of elastic solids under dynamic loading is of practical interest. Thus, investigations of fracture effectiveness of solids by impact loading are directly connected to energetics of the process. Knowledge of energetic processes going on within solid media is also important for dynamic strength testing of materials. For example, in the experiments for impact toughness the nature of energy exchange between the impacter and the sample significantly affects the conclusion about dynamic strength of the material and about energy consumed by fracture process [2–8].

The analysis of the balance of energy for solids undergoing dynamic loading can also be very useful if we think about the problems connected with wave focusing. Experiments on loading of metals and rocks by spherically symmetric waves have shown that in this case it is possible to create spherical void in the ball center. Results of the above mentioned experiments can be found for example in ref. [9–11], and an attempt of interpretation of the received experimental data is performed in ref. [12]. Focusing of elastic and acoustic waves in the center of a solid sphere was also studied in refs. [6,13].

In this paper an analysis of internal energy under nonstationary oscillations of an elastic and acoustic solid sphere is performed. The solution is given as a sum of incident and reflected waves (in D'Alambert form). The influence of pulsed load duration on the behaviour of potential, kinetic and full internal energy of a solid sphere for different times is investigated.

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## 1 Radial oscillations of an acoustic solid sphere

Suppose spherically symmetric radial oscillations of an acoustic solid sphere under uniform surface loading by a load of a given time profile. In spherical coordinate system ($r$, $\theta$, $\eta$) we introduce the following dimensionless parameters:

$$r=\frac{r'}{R_0};\ t=\frac{ct'}{R_0};\ u_r=\frac{u'_r}{R_0};\ \sigma=\frac{\sigma'}{\rho c^2};\quad \varphi=\frac{\varphi'}{cR_0};\ v_r=\frac{v'_r}{c}; \tag{1}$$

where $u_r$ is the displacement, $\sigma$ is the pressure, $\varphi$ is the velocity potential, $t$ is the dimensionless time, $c$ is the sound speed, $\sigma_0$ is the loading constant, $\rho$ is the media mass density, $R_0$ is the ball radius, and $v_r$ is the velocity of particles displacement. Parameters with dimension are marked by "'".

The following initial boundary value problem is to be solved: solution of the equation is to be found

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{2}{r}\frac{\partial\varphi}{\partial r}=\frac{\partial^2\varphi}{\partial t^2}, \tag{2}$$

with the following boundary conditions:

$$\sigma(t)=-\left.\frac{\partial\varphi}{\partial t}\right|_{r=1}=f(t),\ \lim_{r\to0} r^2u_r=\lim_{r\to0} r^2\frac{\partial\varphi}{\partial r}=0 \tag{3}$$

and initial conditions:

$$\varphi=\left.\frac{\partial\varphi}{\partial t}\right|_{t=0}=0, \tag{4}$$

here $f(t)$ is a load given a priori. The second condition in eq. (3) means the absence of source in the center of the ball.

Pressure and velocity inside the ball can be found as:

$$\sigma=-\frac{\partial\varphi}{\partial t},\ v_r=\frac{\partial\varphi}{\partial r}, \tag{5}$$

and the full dimensionless energy of the solid sphere $\varepsilon$ can be found as:

$$E=\int_0^1\left[\left(\frac{\partial\varphi}{\partial t}\right)^2+\left(\frac{\partial\varphi}{\partial r}\right)^2\right]r^2\mathrm{d}r, \tag{6}$$

Laplace transform with parameter $p$ ($L$ is the transformant) can be applied to eqs. (2)–(4):

$$\frac{\partial^2\varphi^L}{\partial r^2}+\frac{2}{r}\frac{\partial\varphi^L}{\partial r}=p^2\varphi^L, \tag{7}$$

$$\sigma^L=-\left.\frac{\partial\varphi^L}{\partial t}\right|_{r=1}=f^L(p),\ \lim_{r\to0} r^2u^L_r=\lim_{r\to0} r^2\frac{\partial\varphi^L}{\partial r}=0, \tag{8}$$

$$\varphi^L=-p\varphi^L\big|_{t=0}=0. \tag{9}$$

The general solution to eq. (7) can be found as:

$$\varphi^L(r,p)=A(p)\cdot\frac{\mathrm{e}^{-pr}}{r}+B(p)\cdot\frac{\mathrm{e}^{pr}}{r}.$$

Integration constants $A$ and $B$ can be found out of transformed boundary conditions eq. (8):

$$A(p)=-\frac{f^L(p)}{p(\mathrm{e}^{-p}-\mathrm{e}^{p})},\ B=-A,$$

then in the transformed domain the solution will take the following form:

$$\varphi^L(r,p)=\frac{1}{r}\cdot\frac{\mathrm{e}^{+pr}-\mathrm{e}^{-pr}}{p(\mathrm{e}^{-p}-\mathrm{e}^{p})}\ f^L(p). \tag{10}$$

Let the load have a rectangular time shape:

$$f(t)=H(t)-H(t-T), \tag{11}$$

where $H(t)$ is the Heaviside step function and $T$ is the load duration. Then $f^L(p)=1-\mathrm{e}^{-pT}/p$. Original eq. (10) can be rewritten as a difference of two integrals:

$$\varphi(r,t)=\frac{1}{2\pi i}\int_L\Phi(p)\mathrm{d}p-\frac{1}{2\pi i}\int_L\Phi(p)\mathrm{e}^{-pT}\mathrm{d}p,\quad \Phi(p)=F(p)\mathrm{e}^{pT},\quad F(p)=\frac{1}{r}\cdot\frac{\mathrm{e}^{pr}-\mathrm{e}^{-pr}}{p^2\left(\mathrm{e}^{-p}-\mathrm{e}^{p}\right)}, \tag{12}$$

where $L$ is the Mellin contour $(\lambda-\mathrm{i}\infty,\lambda+\mathrm{i}\infty)$, placed parallel to imaginary axis of the complex plane $p$ to the right of the axis at a distance of $\lambda$.

Consider first integral in ref. (12). The integrand $F(p)$ in eq. (12) can be rewritten as:

$$F(p)=\frac{1}{r}\cdot\frac{\mathrm{e}^{-p(r+1)}-\mathrm{e}^{p(r-1)}}{p(1-\mathrm{e}^{-2p})}\ f^L(p), \tag{13}$$

by taking into account that Re($p$)=0, the inverse transformation can be performed as a series of expansion in powers of exponent

$$\left[1-\exp(-2p)\right]^{-1}=\sum_{n=0}^{\infty}\exp(-2np), \tag{14}$$

with the following termwise inversion of the received equiconvergent series taking into account the shift theorem. Similarly the second integral in eq. (12) can be calculated. As a result, for rectangularly shaped load with duration $T$ eq. (11) one can receive:

$$\varphi(r,t)=-\frac{1}{r}\sum_{n=1}^{\infty}\Big[\big(t+r-(2n-1)\big)H\big(t+r-(2n-1)\big)$$

$$-\left(t-r-(2n-1)\right)H\left(t-r-(2n-1)\right)$$
$$-\left(t+r-T-(2n-1)\right)H\left(t+r-T-(2n-1)\right)$$
$$+\left(t-r-T-(2n-1)\right)H\left(t-r-T-(2n-1)\right)\Big]. \quad (15)$$

The corresponding pressure $\sigma$ and velocity of the ball particles $v_r$ can be found using eq. (5):

$$\sigma(r,t)=\frac{1}{r}\sum_{n=1}^{\infty}\big[H(t+r-(2n-1))-H(t-r-(2n-1)) - H(t+r-T-(2n-1)) + H(t-r-T-(2n-1))\big]; \quad (16)$$

$$v_r(r,t)=\frac{1}{r^2}\sum_{n=1}^{\infty}\big[(t-(2n-1))(H(t+r-(2n-1)) - H(t-r-(2n-1)))-(t-T-(2n-1)) \times(H(t+r-T-(2n-1)) - H(t-r-T-(2n-1)))\big]. \quad (17)$$

Integrating eq. (17) by $t$, one can find displacements within the ball $u_r$

$$u_r(r,t)=\frac{1}{2r^2}\sum_{n=1}^{\infty}\big[(t+r-(2n-1))(t-r-(2n-1)) \times(H(t+r-(2n-1))-H(t-r-(2n-1))) -(t+r-T-(2n-1))(t-r-T-(2n-1)) \times(H(t+r-T-(2n-1)) - H(t-r-T-(2n-1)))\big]. \quad (18)$$

Based on eq. (17), flux through the spheric surface surrounding the center of the ball is vanishing as the spheric surface radius is approaching zero

$$\lim_{r\to 0}4\pi r^2\frac{\partial\varphi(r,t)}{\partial r}=0, \quad (19)$$

thus, there is no disturbance in the center of the solid sphere. The solution of the problem can also be received as an expansion series for eigenfunctions.

Now it is possible to evaluate the energy inside the solid sphere. To start with, consider the case of infinitely long load pulse $T\to\infty$. The evaluation of potential energy $\Pi$ is not complicated. In order to evaluate the kinetic energy $K$ one can consider first converging and the first diverging wave.

In the interval $0\leqslant t\leqslant 1$

$$K=\int_0^1 v_r^2r^2\mathrm{d}r=\left(t-1\right)^2\int_0^1\frac{1}{r^2}H\left(t+r-1\right)\,\mathrm{d}r.$$

Then integration by parts can be applied

$$K=-(t-1)^2\left[H\left(t\right)-\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}H\left(t+\varepsilon-1\right) -\int_0^1\frac{1}{r}\delta\left(t+r-1\right)\,\mathrm{d}r\right].$$

Using the filtering property of the $\delta$-function

$$\int_a^b f(x)\delta(x-c)\,\mathrm{d}x=f(c)H(c-a)H(b-c). \quad (20)$$

So far in the considered time interval $H(t)$=1, thus

$$K=-\left(t-1\right)^2\left[1-\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}H\left(t+\varepsilon-1\right)-\frac{1}{1-t}\right]$$
$$=-\left(t-1\right)^2\left[-\frac{t}{1-t}-\lim_{\varepsilon\to 0}\delta\left(t+\varepsilon-1\right)\right]$$
$$=-t(t-1)+(t-1)^2\delta(t-1).$$

The last summand is equal to zero, thus

$$K=t(t-1),\quad 0\leqslant t\leqslant 1.$$

In the interval $1\leqslant t\leqslant 2$:

$$K=(t-1)^2\int_0^1\frac{1}{r^2}\big[1-H(t-r-1)\big]\mathrm{d}r$$
$$=\left(t-1\right)^2\left[-1+\lim_{\varepsilon\to 0}\frac{1-H\left(t+\varepsilon-1\right)}{\varepsilon} +\int_0^1\frac{1}{r}\delta\left[-r\left(-\left(t-1\right)\right)\right]\,\mathrm{d}r\right].$$

Taking into consideration the evenness of $\delta$-function one can receive

$$K=(t-1)(2-t),\quad 1\leqslant t\leqslant 2.$$

Similarly the computations can be performed for the resting time intervals.

Potential $\Pi=\Pi(t)$, kinetic $K=K(t)$ and full internal energy $E(t)=\Pi(t)+K(t)$ of the solid sphere as a function of time is given in Figure 1. It is remarkable that the kinetic energy during the time interval from 0 to 2 twice growsto maximum value and then vanishes while the potential energy does only once. The kinetic energy is maximized when the wave front is passing the $r$=0.5. At this moment the ratio between surfaces of the outer spheric layer ($0.5\leqslant r\leqslant 1.0$) and the internal solid sphere ($0\leqslant r\leqslant 0.5$) is equal to 4 and the ratio of the corresponding volumes is 8.

It can be seen from Figure 1 that full internal energy $\varepsilon$ is periodically reduced to zero whereas pressure on the ball surface remains uniform and continuous. Such a situation is essentially different from the corresponding static problem. This behaviour of energy seems to be paradoxical from the first sight but can be easily explained. Constant pressure on the ball surface is kept by external forces potential that is created by a loading device. Energy exchange between the ball and the loading device leads to periodical disappearance of the full internal energy in the ball. It can also be noticed that at the moments of time when the internal energy within

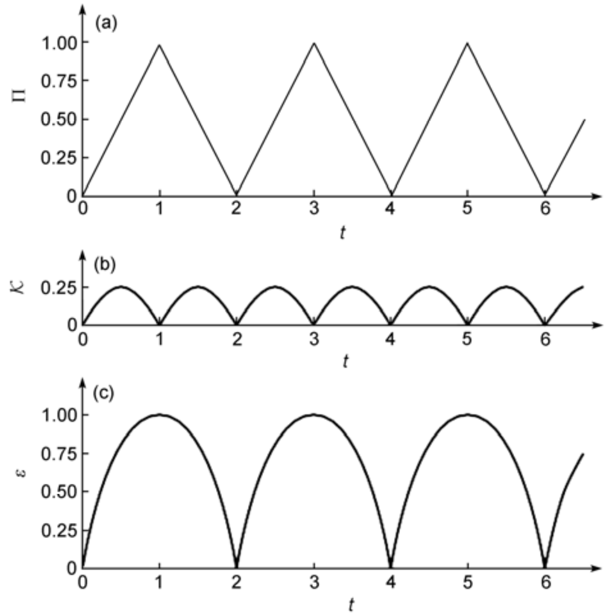


**Figure 1** Energy of an acoustic ball as a function of time (a) Potential (Π), (b) kinetic (*K*) and (c) full (*ε*), $T\to\infty$.

the ball equals to zero, all the points of the ball are at rest. Similar effects were investigated in refs. [1,2]. The received result demonstrates that dynamically loaded objects even in simple cases are not closed systems in thermo-dynamical sense.

Consider the case when the load has finite duration. Let, for example $T$=0.2. Evaluation of potential $\Pi$ and kinetic $K$energy in this case is not significantly different from the case of infinite load duration $T\to\infty$. The only difference is that in this case time intervals for integration should be less.

Here an example of calculations for the few first time intervals is given. As before, integration by parts is used and evenness of $\delta$-function is utilized and eq. (20) is also used.

In this case of the finite load duration the load can be received as a sum of the initial compressive wave, and a tensile wave with the onset shifted by time $T$ as compared to the onset of the initial wave. Then

$$\Pi=\int_0^1\left[\sigma(t,r)-\sigma(t-T,r)\right]^2 r^2\mathrm{d}r.$$

For time interval $0\leqslant t\leqslant 2$

$$\Pi=\begin{cases} t, & 0\leqslant t\leqslant T,\\ T, & T\leqslant t\leqslant 1,\\ 2(1-t)+T, & 1\leqslant t\leqslant 1+T/2,\\ 2(t-1)-T, & 1+T/2\leqslant t\leqslant 1+T,\\ T, & 1+T\leqslant t\leqslant 2,\end{cases}$$

The kinetic energy can be received as:

$$K=\int_0^1\left(v_r(t,r)-v_r(t-T,r)\right)^2 r^2\mathrm{d}r.$$

After calculations it is possible to receive

$$K=\begin{cases} t(1-t), & 0\leqslant t\leqslant T,\\ T(1-T), & T\leqslant t\leqslant 1,\\ 2(t-1)+T(1-T), & 1\leqslant t\leqslant 1+T/2,\\ 2(1-t)+T(3-T), & 1+T/2\leqslant t\leqslant 1+T,\\ T(1-T), & 1+T\leqslant t\leqslant 2.\end{cases}$$

Received potential and kinetic energy time dependencies are given in Figure 2. Behaviour of potential and kinetic energy is rather clear from analysis of diagrams for pressure and velocity. Maximal and minimal values for $\Pi$ and $K$ are placed in *antiphase*, and thus the full internal energy $\varepsilon$ of the solid sphere for $t$>0.2 is unchanged. Similar diagrams can be plotted for arbitrary load duration.

## 2 Radial oscillations of an elastic sphere

Consider an elastic homogeneous isotropic media having a form of a solid sphere with radius $R$. Spherical coordinates ($r$, $\theta$, $\eta$) with the origin in the ball center are introduced. Let $\Phi$ be the wave potential. In conditions of spheric symmetry the points of the media satisfy the following condition:

$$\frac{\partial^2\Phi}{\partial t^2}=\frac{c_1^{\,2}}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right). \tag{21}$$

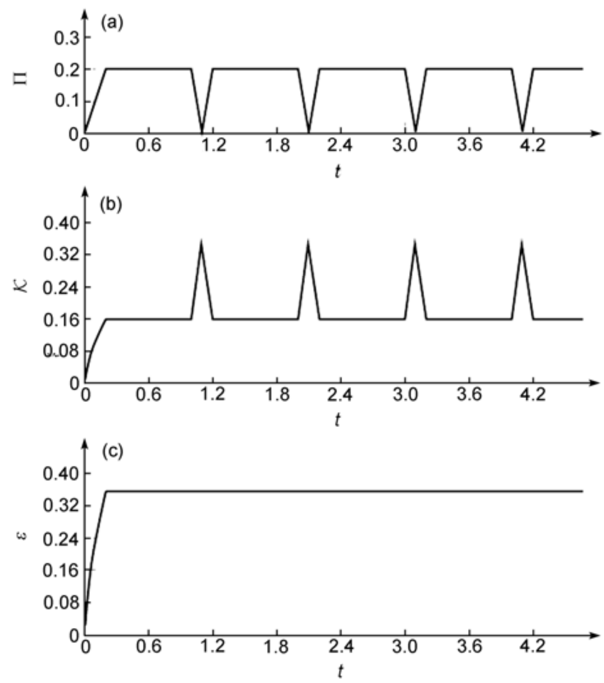


**Figure 2** Energy of an acoustic ball as a function of time (a) Potential (Π), (b) kinetic (*K*) and (c) full (*ε*), *T*=0.2.

Here $c_1$ is the longitudinal wave speed. Let $u$ be the displacement vector. Components of this vector satisfy the following conditions

$$u_r = u(t,r), \quad u_\theta \equiv u_\eta \equiv 0, \quad u = \frac{\partial \Phi}{\partial r}. \tag{22}$$

Suppose that for all moments of time prior to loading onset (for $t<0$) all the points of the solid sphere are in rest and stresses are zero in the ball:

$$\Phi\big|_{t<0} = 0, \quad \frac{\partial \Phi}{\partial t}\bigg|_{t<0} = 0. \tag{23}$$

At $t=0$ the boundary of the ball $r=R$ is loaded by a stress:

$$\sigma\big|_{r=R} = f(t). \tag{24}$$

At the same time in the center of the solid sphere in view of the central symmetry, the following condition should be fulfilled:

$$u = \frac{\partial \Phi}{\partial r}\bigg|_{r=0} = 0. \tag{25}$$

The solvability of the problem requires that eqs. (21)–(25) should be supplemented by interconnection between stresses and displacements. Let $c_2$ be the speed of the transversal wave and $\gamma=c_2/c_1$. Then

$$\sigma_r = \rho \cdot c_1^2 \left[\frac{\partial u}{\partial r} + 2\left(1-2\gamma^2\right)\frac{u}{r}\right]. \tag{26}$$

Stress $\sigma_r$ can be found via wave potential:

$$\sigma_r = \rho \cdot c_1^2 \left[\frac{1}{c_1^2}\frac{\partial^2 \Phi}{\partial t^2} - \frac{4\gamma^2}{r}\frac{\partial \Phi}{\partial r}\right]. \tag{27}$$

The solution can be presented in D'Alambert form. In order to do so function $\Psi(t,r) = r \cdot \Phi(t,r)$ should be introduced. Then

$$u = \frac{\partial \Phi}{\partial r} = \frac{\partial}{\partial r}\left(\frac{r\Phi}{r}\right) = \frac{1}{r}\cdot\frac{\partial \Psi}{\partial r} - \frac{1}{r^2}\Psi,$$

$$\frac{\partial}{\partial r}\left(\frac{r^2 \partial \Phi}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{r\partial \Psi}{\partial r} - \Psi\right) = r \cdot \frac{\partial^2 \Psi}{\partial r^2}$$

and eq. (21) is transformed into wave equation for $\Psi$:

$$c_1^2 \frac{\partial^2 \Psi}{\partial r^2} = \frac{\partial^2 \Psi}{\partial t^2}. \tag{28}$$

Radial stress can be now presented as:

$$\sigma_r = \frac{\rho \cdot c_1^2}{r}\left[\frac{\partial^2 \Psi}{\partial r^2} - \frac{4\gamma^2}{r}\frac{\partial \Psi}{\partial r} + \frac{4\gamma^2}{r^2}\Psi\right]. \tag{29}$$

Solution of eq. (28) for times $t<R/c_1$ will be given by the wave moving towards the center of the ball i.e. $\Psi(t, r)=\Psi(t+(r-R)/c_1)$ Thus

$$\frac{\partial \Psi}{\partial r} = \frac{\Psi'}{c_1}, \quad \frac{\partial^2 \Psi}{\partial r^2} = \frac{\Psi''}{c_1^2}.$$

Eq. (29) is providing a connection between stresses and function $\Psi$ for all permissible $r$. On the boundary $r = R$:

$$\sigma_r = \frac{\rho \cdot c_1^2}{R}\left[\frac{\Psi''}{c_1^2} - \frac{4\gamma^2 \cdot \Psi'}{c_1 R} + \frac{4\gamma^2}{R^2}\Psi\right].$$

We denote $2\gamma \cdot c_1 / R = a$ for the sake of brevity. Then

$$\frac{\sigma_r \cdot R}{\rho} = \Psi'' - 2a\gamma\Psi' + a^2\Psi.$$

On the boundary we assume $r = R$ at $t = 0$, $\Phi = 0$ and $u = 0$, which is equivalent to zero initial conditions for function $\Psi$ and its derivative. The solution to initial boundary problem:

$$\Psi'' - 2a\gamma \cdot \Psi' + a^2\Psi = \frac{f(t)\cdot R}{\rho}, \tag{30}$$

$$\Psi\big|_{t=0} = 0, \quad \Psi'\big|_{t=0} = 0, \tag{31}$$

can be presented as:

$$\Psi(t) = k_1(t)\cdot w_1(t) + k_2(t)\cdot w_2(t). \tag{32}$$

For the load $f(t)=P_0H(t)$, where $H(t)$ is the Heaviside step function one can receive

$$\begin{aligned} w_1(t) &= \exp(a\gamma t)\sin\left(a\sqrt{1-\gamma^2}t\right)H(t), \\ w_2(t) &= \exp(a\gamma t)\cos\left(a\sqrt{1-\gamma^2}t\right)H(t), \end{aligned} \tag{33}$$

$$\begin{aligned} k_1(t) = &-\frac{P_0 \cdot R}{\rho}\cdot\frac{\exp(-2\gamma \cdot at)}{a^2\sqrt{1-\gamma^2}} \\ &\times\left[\gamma w_2(t) - \sqrt{1-\gamma^2}w_1(t)\right] + l_1, \end{aligned} \tag{34}$$

$$\begin{aligned} k_2(t) = &\frac{P_0 \cdot R}{\rho}\cdot\frac{\exp(-2\gamma \cdot at)}{a^2\sqrt{1-\gamma^2}} \\ &\times\left[\gamma w_1(t) + \sqrt{1-\gamma^2}w_2(t)\right] + l_2, \end{aligned} \tag{35}$$

where constants $l_1$ and $l_2$ can be found from initial conditions (31). Substituting eqs. (33), (34) into eq. (32) after simplifications one can receive

$$\Psi(t) = \frac{P_0 \cdot R}{\rho \cdot a^2} + l_1 \cdot w_1(t) + l_2 \cdot w_2(t).$$

Thus, $\Psi(0)=P_0\cdot R/(\rho\cdot a^2)+l_2$ and $l_2=-P_0\cdot R/(\rho\cdot a^2)$. Then

$$\Psi'(t)=l_1\cdot a\cdot\left[\gamma\cdot w_1(t)+\sqrt{1-\gamma^2}w_2(t)\right]-\frac{P_0\cdot R}{\rho\cdot a}\cdot\left[\sqrt{1-\gamma^2}w_2(t)-\gamma\cdot w_1(t)\right].$$

And

$$\Psi'(0)=l_1\cdot a\cdot\sqrt{1-\gamma^2}-P_0\cdot R\cdot\gamma/(\rho\cdot a),$$
$$l_1=P_0\cdot R\cdot\gamma\Big/\left(\rho\cdot a\cdot\sqrt{1-\gamma^2}\right).$$

Thus,

$$\Psi(t)=\frac{P_0\cdot R}{\rho\cdot a^2}\cdot\left[1+\frac{\gamma}{\sqrt{1-\gamma^2}}\cdot w_1(t)-w_2(t)\right]. \quad (36)$$

Returning to $\Phi(t,r)=\Psi(t,r)/r$, one can receive potential for the wave moving to the ball center

$$\Phi_{(0)}(t,r)=\frac{P_0\cdot R^3}{4\rho c_1^2\gamma^2\cdot r}\left(1-w_2\left(t+\frac{r-R}{c_1}\right)+\frac{\gamma}{\sqrt{1-\gamma^2}}\cdot w_1\left(t+\frac{r-R}{c_1}\right)\right). \quad (37)$$

This formula will provide a solution for eqs. (21)–(25) up to the moment when the wave front will reach the ball center $t<R/c_1$. Displacements and radial stresses in this wave can be found as:

$$u_{(0)}(t,r)=\frac{P_0\cdot R^2}{4\rho c_1^2\gamma^2 r}\left(-\frac{R}{r}+\frac{R}{r}w_2\left(t+\frac{r-R}{c_1}\right)+\frac{\gamma}{\sqrt{1-\gamma^2}}\left(2-\frac{R}{r}\right)w_1\left(t+\frac{r-R}{c_1}\right)\right),$$

$$\sigma_{r(0)}(t,r)=P_0\frac{R}{r}\left(\frac{R^2}{r^2}+\left(1-\frac{R^2}{r^2}\right)w_2\left(t+\frac{r-R}{c_1}\right)+\frac{\gamma}{\sqrt{1-\gamma^2}}\left(1-\frac{R}{r}\right)^2 w_1\left(t+\frac{r-R}{c_1}\right)\right).$$

For $t\geqslant R/c_1$ the wave moving from the boundary $r=R$ is interacting with the center of the ball $r=0$ and is producing a wave moving from the ball center. Function $\Psi$ for this wave will take the following form

$$\Psi_{(1)}=A_1\cdot\Psi\left(t-\frac{r+R}{c_1}\right), \quad (38)$$

where unknown $A_1$ can be found from boundary condition eq. (25). Thus

$$u(t,r)=u_{(0)}(t,r)+u_{(1)}(t,r).$$

Value for displacement $u$ in the reflected wave can be found as:

$$u_{(1)}(t,r)=A_1\left(\frac{1}{r^2}\cdot\Psi+\frac{1}{c_1\cdot r}\cdot\Psi'\right).$$

In accordance with eq. (25), we should request that $u_{(0)}(t,0)+u_{(1)}(t,0)=0$. Suppose that the front of the reflected wave had approached a point with coordinate $r_0$. Then

$$t-R/c_1=r_0/c_1,\ t-(r+R)/c_1=(r_0-r)/c_1,$$
$$t+(r-R)/c_1=(r_0+r)/c_1.$$

As a result one can receive

$$u_{(0)}(t,r)+u_{(1)}(t,r)=\frac{P_0\cdot R^3}{4\rho\cdot r^2\gamma^2c_1^2}\cdot\left[w_2\left(\frac{r_0+r}{c_1}\right)+A_1w_2\left(\frac{r_0-r}{c_1}\right)-1-A_1-\frac{\gamma}{\sqrt{1-\gamma^2}}\left(w_1\left(\frac{r_0+r}{c_1}\right)+A_1w_1\left(\frac{r_0-r}{c_1}\right)\right)+\frac{2\gamma r}{R\sqrt{1-\gamma^2}}\cdot\left(w_1\left(\frac{r_0+r}{c_1}\right)-A_1w_1\left(\frac{r_0-r}{c_1}\right)\right)\right].$$

It is possible to expand functions $w_1$ and $w_2$ from this expression into series by powers of $r$ in the vicinity of $r$=0:

$$u_{(0)}(t,r)+u_{(1)}(t,r)\underset{r\to 0}{\to}\frac{P_0\cdot R^3}{4\rho\cdot r^2\gamma^2c_1^2}\cdot\left[(1+A_1)\left(w_2\left(\frac{r_0}{c_1}\right)-1-\frac{\gamma}{\sqrt{1-\gamma^2}}w_1\left(\frac{r_0}{c_1}\right)\right)+2(1+A_1)\frac{\gamma^2}{R^2}\left(\frac{\gamma}{\sqrt{1-\gamma^2}}w_1\left(\frac{r_0}{c_1}\right)+w_2\left(\frac{r_0}{c_1}\right)\right)r^2+O(r^3)\right].$$

Expression inside the square brackets can be equal to zero while $r\to 0$ only if $A_1=-1$. Thus expression for $\Phi_{(1)}$ will be given by

$$\Phi_{(1)}(t,r)=-\frac{P_0\cdot R^3}{4\rho c_1^2\gamma^2\cdot r}\times\left(1-w_2\left(t-\frac{r+R}{c_1}\right)+\frac{\gamma}{\sqrt{1-\gamma^2}}\cdot w_1\left(t-\frac{r+R}{c_1}\right)\right) \quad (39)$$

and the displacement $u$ and the stress $\sigma_r$ can be received as:

$$u_{(1)}(t,r)=\frac{P_0\cdot R^2}{4\rho c_1^2\gamma^2 r}\left(\frac{R}{r}-\frac{R}{r}w_2\left(t-\frac{r+R}{c_1}\right)+\frac{\gamma}{\sqrt{1-\gamma^2}}\left(2+\frac{R}{r}\right)w_1\left(t-\frac{r+R}{c_1}\right)\right). \quad (40)$$

$$\sigma_{r(1)}(t,r) = -P_0 \frac{R}{r}\left(\frac{R^2}{r^2} + \left(1-\frac{R^2}{r^2}\right) w_2\left(t-\frac{r+R}{c_1}\right) + \frac{\gamma}{\sqrt{1-\gamma^2}}\left(1+\frac{R}{r}\right)^2 w_1\left(t-\frac{r+R}{c_1}\right)\right). \quad (41)$$

Moving towards the boundary $r = R$ the reflected wave will approach it at $t=R/c_1$ and will start reflecting from it. This newly reflected wave (denoted with subscript 2), should be such, that the condition (2.4) is satisfied on the solid sphere boundary. This condition is already fulfilled by the wave with subscript 0. Thus, on the solid sphere boundary sum of stresses within the waves with subscripts 1 and 2 should be zero. From eq. (41) one can receive

$$\sigma_{r(2)}(t,R) = -\sigma_{r(1)}(t,R) = P_0\left(1+\frac{4\gamma}{\sqrt{1-\gamma^2}} w_1\left(t-\frac{2R}{c_1}\right)\right). \quad (42)$$

For the wave with subscript 2 moving towards the ball center for times $t \geqslant 2R/c_1$ function $\Psi$, as a solution of wave equation (28), will take the following form: $\Psi(t, r)= \Psi(t+(r-3R)/c_1)$. In order to find $\Psi$ for this wave system (30) and (31) should be solved again. As for function $f$ in eq. (30), one should use an expression from the right-hand side of eq. (42) and as the initial moment of time in eq. (31) one should take $t=R/c_1$. Finding the solution analogues to above, one can receive

$$\Psi_{(2)}(t) = \Psi_{(0)}(t) + \frac{P_0 \cdot R^2}{\rho c_1^2 \gamma\left(1-\gamma^2\right)} \times\left[\frac{R}{2\sqrt{1-\gamma^2}} w_1(t) - \gamma \cdot c_1 t \cdot w_2(t)\right].$$

Substituting this into $\Phi(t, r)=\Psi(t+(r-3R)/c_1)/r$ the wave $\Phi_2$ can be found as:

$$\Phi_{(2)}(t,r) = \Phi_{(0)}\left(t-\frac{2R}{c_1}, r\right) + \frac{P_0 \cdot R^3}{2\rho c_1^2 \gamma\left(1-\gamma^2\right)\sqrt{1-\gamma^2}\cdot r} \times w_1\left(t+\frac{r-3R}{c_1}\right) - \frac{P_0 \cdot R^2 \cdot \left(c_1 t + r - 3R\right)}{\rho c_1^2\left(1-\gamma^2\right)\cdot r} \times w_2\left(t+\frac{r-3R}{c_1}\right). \quad (43)$$

Then, satisfying condition eq. (25) in the solid sphere center for $\Psi_2$ using $\Psi_3$, wave with subscript 3 can be found. Continuation of this process provides a possibility to receive all the consequent waves.

Consider arbitrary load $f(t)$. Wave potential for this problem can be found using time convolution (Duhamel's integral) of the solution received above for the load $H(t)$. Thus, for $t \geqslant t_n$

$$\Phi_n^{(f)}(t,r) = \int_{t_n}^{t} \Phi_n^{(H)}(s,r)\cdot f'(t-s)\,\mathrm{d}s, \quad (44)$$

where $t_n$ is the time when the corresponding wave approaches the point with coordinate $r$. For the waves with even subscripts $t_n=(R\cdot(2n+1)-r)/c_1$, and for the waves with odd subscripts $t_n=(R\cdot(2n+1)+r)/c_1$. For $t<t_n$ potential is equal to zero.

Now we examine the behaviour of energy in the elastic problem.

Kinetic energy of the ball in the considered case of single non-zero displacement component can be found as:

$$K(t) = \frac{1}{2}\rho\iiint_\Omega \left(\frac{\partial u}{\partial t}(t,r)\right)^2 \mathrm{d}\Omega = 2\pi\rho\int_0^R \left(\frac{\partial u}{\partial t}(t,r)\right)^2 r^2 \mathrm{d}r. \quad (45)$$

Kinetic energy for the load given by $H(t)$ for different $\gamma$ is presented in Figure 3(a). Kinetic energy time dependence is non-monotonic. Time intervals between minimal and maximal values are approximately equal, but are changing as the vale for $\gamma$ is changed. For small $\gamma$ kinetic energies of the solid sphere for elastic and acoustic problems are not significantly different. Potential energy of deformation, expressed through components of tensor of stresses $\sigma_{ij}$ and tensor of strains $\varepsilon_{ij}$, can be calculated as:

$$\Pi(t) = \frac{1}{2}\iiint_\Omega \sigma_{ij}\varepsilon_{ij}\mathrm{d}\Omega.$$

Apart fromwhat is derived above $\sigma_r$ nonzero components of the stresses tensor will also include $\sigma_\theta$ and $\sigma_\eta$. Values for these components can be found from displacements:

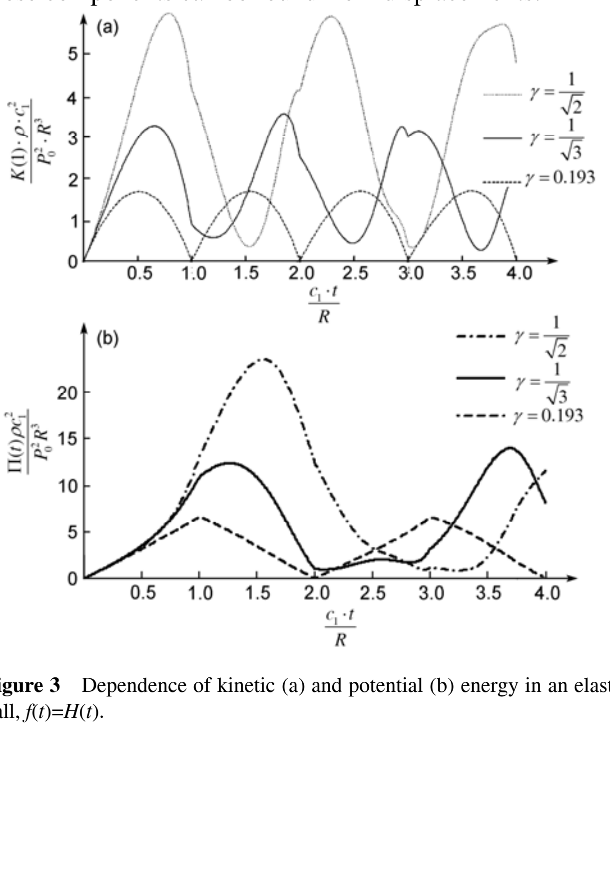


**Figure 3** Dependence of kinetic (a) and potential (b) energy in an elastic ball, $f(t)=H(t)$.

$$\sigma_\theta = \sigma_\eta = \rho \cdot c_1^2 \left[ \left(1-2\gamma^2\right)\frac{\partial u}{\partial r} + 2\left(1-\gamma^2\right)\frac{u}{r} \right].$$

Nonzero components of the strains tensor can be found as:

$$\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \varepsilon_\eta = \frac{u}{r}.$$

Thus, potential energy of deformation of the ball can be received as:

$$\Pi(t) = \frac{1}{2}\iiint_\Omega (\sigma_r\varepsilon_r + \sigma_\theta\varepsilon_\theta + \sigma_\eta\varepsilon_\eta)\mathrm{d}\Omega = 2\pi \cdot \int_0^R (\sigma_r\varepsilon_r + \sigma_\theta\varepsilon_\theta + \sigma_\eta\varepsilon_\eta) r^2 \mathrm{d}r. \quad (46)$$

Calculations of the potential energy for the load given by $H(t)$ for different values of $\gamma$ are presented in Figure 2(b). Behaviour of the potential energy as a function of time is non-monotonicand this is for kinetic energy. However extremes for kinetic and potential energy are not coinciding in time. Maximal value for potential energy is about 4 times more as compared to kinetic energy. Besides that, for some times value for kinetic energy can exceed the corresponding value for the potential energy. Full internal energy is equal to the work spent in order to displace the solid sp here boundary by the applied force $P_0$, i.e.

$$K(t) + \Pi(t) = 4\pi \cdot R^2 \cdot P_0 \cdot u(t, R). \quad (47)$$

Figure 4 gives the full internal energy of the solid sphere as a function of time for different values of $\gamma$ for the load $f(t)=H(t)$. In contrast to the acoustic case, displacement is not vanishing at the moment when the wave front is returning to the boundary ($t=2R/c_1$). For small $\gamma$ displacement is close to the one observed in theacoustic case.It is essential that the full energy is oscillating and for small values of $\gamma$ the value of internal energy is periodically dropping almost to zero.

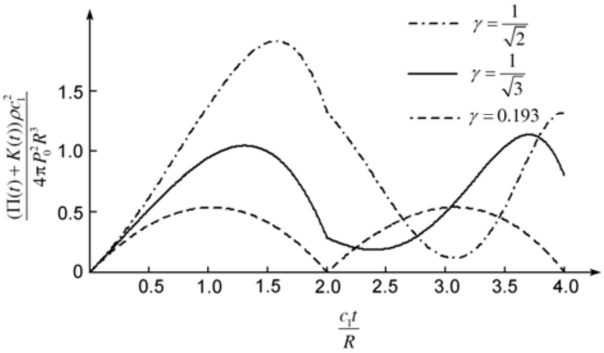


**Figure 4** Full internal energy of an elastic ball, $f(t)=H(t)$.

## 3 Conclusions

The received analytical solutions to two problems for dynamically loaded acoustic and elastic solid sphere illustrate an important peculiarity of dynamic deformation consisting in a specific behavior of energy. The analysis demonstrates that in the case of dynamic loading internal energy of a body is significantly time dependent and can periodically vanish. This fact seems to be strange at first sight. This effect is fundamental for dynamics of deformation and is caused by a possibility of energy exchange between the loaded body and the external loading device, creating a corresponding external force potential.

Investigation results are the evidence of necessity to account for energetical peculiarities when materials for dynamic strength are tested (for example using Charpy method). Such experiments should be analyzed with account of energy exchange between the sample and the loading device. In this connection the importance of experimental schemes in which material fracture is initiated only after removal of external loads [1,3] is evident. This type of experimental techniques provides a possibility for correct estimation of full internal energy of a body at the moment of dynamic fracture, which, in the final analysis, makes it possible for correct estimation of energy spent for material fracture.

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