Proceedings of the International Conference DAYS on DIFFRACTION 2016, pp. 84–88 

# On steady-state moving load problems for an elastic half-space

V. Bratov
Department of Elasticity, Saint-Petersburg University, Russia; e-mail: vladimir.bratov@gmail.com
J. Kaplunov, D.A. Prikazchikov
School of Computing and Mathematics, Keele University, United Kingdom;
e-mail: j.kaplunov@keele.ac.uk, d.prikazchikov@keele.ac.uk

**The steady-state regime of a moving load on an elastic half-plane is addressed. It is shown that the solution can be expressed through a single harmonic function, similarly to the known eigensolution for surface Rayleigh wave, thus reducing a vector problem in linear elasticity to a scalar one for the Laplace equation. Examples of steadily moving vertical force and punch are investigated, illustrating the proposed approach.**

## 1 Introduction

The problem of a moving load on elastic half-space has been studied since well-known paper [1], followed by a number of contributions, see e.g. [2] and references therein. We also mention the work [3], revisiting the results of [1] in the transonic range. It is remarkable that the steady-state solution is defined up to rigid body motion components, with those determined from the corresponding transient problem [4].

It is known since [1] that resonant behaviour is associated with the case when the speed of the load coincides with the Rayleigh wave speed. A progress in investigating the near-resonant regimes has been reported in [5], within the framework of the explicit model for the Rayleigh wave [6], see also [7] and references therein. This approach has been recently developed to 3D moving loads, allowing simple approximate solutions in terms of elementary functions, see [8] and [9].

The aforementioned asymptotic model for the Rayleigh wave is relying on the representation of the surface wave field in terms of a single plane harmonic function derived in [10], see also [11]. It is interesting that a similar representation was obtained earlier in [12]. Unfortunately, it went seemingly unnoticed by the "western" academic community.

The idea of this article originates from an analogy of the problem under consideration with the aforementioned representation for the Rayleigh wave field of general time-dependence. Below the subsonic solution of the steady-state moving load problems is also obtained in the form of plane harmonic functions.

The paper is organised as follows. First of all, we state that the wave potentials satisfy elliptic equations in the moving coordinate system for the subsonic speeds of the moving load. Then, the methodology of [10], reducing a vector problem in 2D elasticity to a scalar problem for the Laplace equation, is developed. Several illustrative examples are presented. Finally, an extension of the technique to moving punch is discussed.

## 2 Statement of the problem for a vertical force

Consider dynamics of an elastic isotropic half-plane $-\infty < x_1 < \infty$, $0 \leqslant x_2 < \infty$, caused by the action of a force $P = P(x_1 - ct)$, moving steadily at a constant speed $c$, see Fig. 1. In what follows, we focus on the subsonic regime $c < c_2 < c_1$, where $c_1$ and $c_2$ are the longitudinal and transverse wave speeds, respectively.

The equations of motion are taken in standard form, e.g. see [13],

$$\sigma_{ij,j} = \rho u_{i,tt}, \quad i, j, = 1, 2, \tag{1}$$

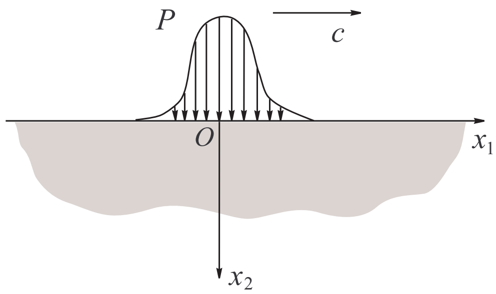


Figure 1: Moving force.

with the Einstein summation convention assumed and comma indicating differentiation along the associated spatial or time variable. Here $\sigma_{ij}$ and $u_i$ denote the components of the stress tensor and displacement vector, respectively, and $\rho$ is the volume density.

The constitutive relations for a linear isotropic solid are given by

$$\sigma_{ij} = \lambda\delta_{ij}u_{k,k} + \mu(u_{i,j} + u_{j,i}), \quad i,j,k = 1,2, \tag{2}$$

where $\delta_{ij}$ is the Kronecker delta, and $\lambda$ and $\mu$ are the Lamé elastic moduli.

The boundary conditions along the surface $x_2 = 0$ are written as

$$\sigma_{12} = 0, \qquad \sigma_{22} = P(x_1 - ct). \tag{3}$$

Let us introduce the elastic wave potentials through

$$u_1 = \phi_{,1} + \psi_{,2}, \qquad u_2 = \phi_{,2} - \psi_{,1}. \tag{4}$$

Then, on employing the Hook's law (2), the original equations (1) become

$$\Delta\phi - c_1^{-2}\phi_{,tt} = 0, \qquad \Delta\psi - c_2^{-2}\psi_{,tt} = 0. \tag{5}$$

Here $\Delta$ is the two-dimensional Laplace operator in variables $x_1$ and $x_2$, and $c_1 = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_2 = \sqrt{\mu/\rho}$ are the longitudinal and transverse wave speed, respectively. The boundary conditions (3) are expressed in terms of the potentials $\phi$ and $\psi$ as follows:

$$2\phi_{,12} + \psi_{,11} - \psi_{,22} = 0, \tag{6}$$

$$\lambda\phi_{,11} + (\lambda + 2\mu)\phi_{,22} + 2\mu\psi_{,12} = P(x_1 - ct). \tag{7}$$

Introducing the moving coordinate $\xi = x_1 - ct$, the governing equations (5) take the form

$$\phi_{,22} + \alpha_1^2\phi_{,\xi\xi} = 0, \qquad \psi_{,22} + \alpha_2^2\psi_{,\xi\xi} = 0, \tag{8}$$

where $\alpha_k^2 = 1 - c^2/c_k^2$ $(k = 1,2)$. Equations (8) are elliptic for subsonic speeds $(c < c_2)$ under consideration. Therefore, the sought for potentials are plane harmonic functions, i.e.

$$\phi = \phi(\xi, \alpha_1 x_2), \qquad \psi = \psi(\xi, \alpha_2 x_2). \tag{9}$$

On substituting the latter into the boundary conditions (7), we have

$$2\phi_{,\xi 2} + (1 + \alpha_2^2)\psi_{,\xi\xi} = 0, \tag{10}$$

$$-(1 + \alpha_2^2)\phi_{,\xi\xi} + 2\psi_{,\xi 2} = \frac{P(\xi)}{\mu}. \tag{11}$$

Noting parallels with the consideration in [10] for the Rayleigh wave and using the Cauchy–Riemann identities for harmonic functions, it is possible to relate the elastic potentials to each other as

$$\psi = \frac{2\alpha_1}{\alpha_2^2 + 1}\phi^*, \tag{12}$$

with the asterisk denoting a harmonic conjugate. Hence, the displacement components are expressed in terms of a single harmonic function as

$$u_1(\xi, x_2) = \phi_{,\xi}(\xi, \alpha_1 x_2) - \frac{2\alpha_1\alpha_2}{1 + \alpha_2^2}\,\phi_{,\xi}(\xi, \alpha_2 x_2), \tag{13}$$

and

$$u_2(\xi, x_2) = \phi_{,2}(\xi, \alpha_1 x_2) - \frac{2}{1 + \alpha_2^2}\,\phi_{,2}(\xi, \alpha_2 x_2). \tag{14}$$

Using the relation (12), the second boundary condition (11) takes the form

$$\phi_{,\xi\xi}\big|_{x_2=0} = -\frac{(1 + \alpha_2^2)\,P(\xi)}{\mu R(c)}, \tag{15}$$

where

$$R(c) = (1 + \alpha_2^2)^2 - 4\alpha_1\alpha_2, \tag{16}$$

confirming a resonance occurring when the speed of the load is equal to that of the Rayleigh wave speed.

The last expression (15) is a boundary condition for the elliptic equation $(8_1)$, say, if $P(\xi) = P_0\dfrac{dp}{d\xi}$, then on employing the Poisson formula, see e. g. [14], the derivative $\phi_{,\xi}$ is given by

$$\phi_{,\xi}(\xi, \alpha_1 x_2) = \frac{A}{\pi R(c)}\int_{-\infty}^{\infty}\frac{\alpha_1 x_2 p(r)}{(r - \xi)^2 + \alpha_2^2 x_2^2}\,dr, \tag{17}$$

enabling a straightforward calculation of the displacement field using (13), (14). In the above,

$$A = -\frac{(1 + \alpha_2^2)\,P_0}{\mu}.$$

For example, for a moving point load $P(\xi) = P_0\delta(\xi)$, e.g. see [1], integration in (15) gives

$$\phi_{,\xi}\big|_{x_2=0} = \frac{A}{R(c)}\left[H(\xi) - \frac{1}{2}\right]. \tag{18}$$

Here an arbitrary constant generally cannot be determined from the steady-state formulation, and may only be obtained from the analysis of the associated transient problem, for more details see [4]

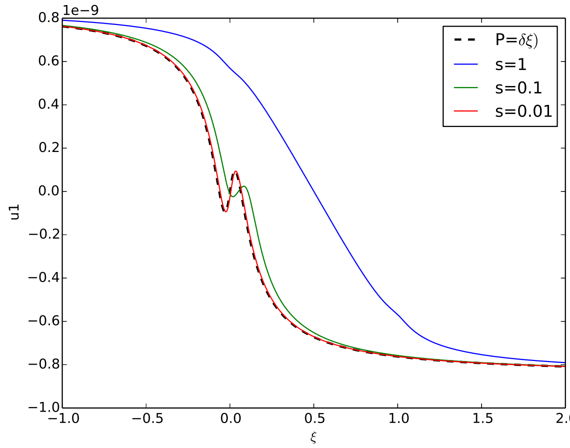


Figure 2: Convergence of the horizontal displacement for a distributed load to solution for point load.

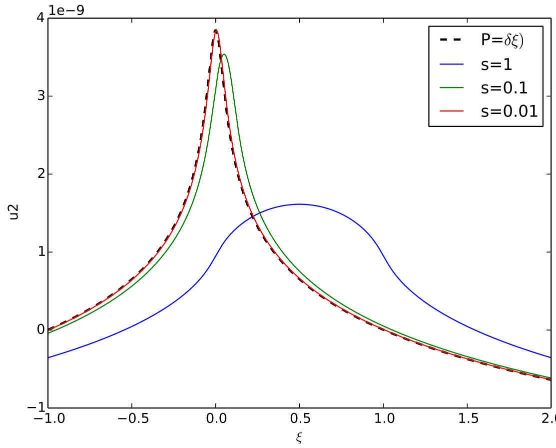


Figure 3: Convergence of the vertical displacement for a distributed load to solution for point load.

and also [5]. The value $\frac{1}{2}$ in (18) is chosen in order to have symmetry.

On satisfying the boundary condition (18), the harmonic function $\phi_{,\xi}(\xi, \alpha_1 x_2)$ is found as

$$\phi_{,\xi}(\xi, \alpha_1 x_2) = \frac{A}{\pi R(c)} \tan^{-1} \frac{\xi}{\alpha_1 x_2}. \qquad (19)$$

Finally, the displacements are given by

$$u_1(\xi, x_2) = \frac{A}{\pi R(c)} \Bigg[ \tan^{-1} \frac{\xi}{\alpha_1 x_2} - \frac{2\alpha_1\alpha_2}{1+\alpha_2^2} \tan^{-1} \frac{\xi}{\alpha_2 x_2} \Bigg] \qquad (20)$$

and

$$u_2(\xi, x_2) = -\frac{\alpha_1 A}{2\pi R(c)} \Bigg[ \ln\big(\xi^2 + \alpha_1^2 x_2^2\big) - \frac{2}{1+\alpha_2^2} \ln\big(\xi^2 + \alpha_2^2 x_2^2\big) \Bigg] \qquad (21)$$

These expressions are identical to those (Eqs. 25, 26) presented in [1], up to a rigid body motion component of the horizontal displacement.

Consider now an example of a distributed load in the shape of a step-function

$$P(\xi) = \frac{H(\xi) - H(\xi - s)}{s}, \qquad (22)$$

where $H(\xi)$ is the Heaviside function and $s$ is the width of a step. It is clear that in the limit $s \to 0$, this step function will become the Dirac delta. The displacements for the distributed load (22) are given by expressions (13) and (14), where the functions $\phi_{,\xi}$ and $\phi_{,2}$ are given by

$$\phi_{,\xi}(\xi, \alpha x_2) = \xi \tan^{-1} \frac{\xi}{\alpha x_2} - \frac{\alpha x_2}{2} \ln\big(\xi^2 + \alpha^2 x_2^2\big) \qquad (23)$$

and

$$\phi_{,2}(\xi, \alpha x_2) = \xi \ln(\xi^2 + \alpha^2 x_2^2) - 2\xi + 2\alpha x_2 \tan^{-1} \frac{\xi}{\alpha x_2}. \qquad (24)$$

The following Figs. 2–5 illustrate the obtained displacement field (20), (21), (23), and (24), scaled by $P_0$. The material parameters used in calculations are those of the PMMA, for which the bulk wave speeds are $c_1 = 2730$ km/h, $c_2 = 1430$ km/h, with the force moving at $c = 0.9c_2 = 1287$ km/h, with the volume density $\rho = 1180$ kg/m$^3$.

Figs. 2 and 3 show the limiting process as the solution for distributed force tends to that of the point force, corresponding to $s \to 0$ for the horizontal and vertical displacements, respectively. The dependence on the moving coordinate $\xi$ is presented, with depth variable set to $x_2 = 0.1$.

In Figs. 4 and 5 the variation of the horizontal and vertical displacements versus the moving coordinate $\xi$ is shown for several values of depth $x_2$. The graphs in Figs. 4 and 5 are plotted for a distributed load of width $s = 1$. The decay of the amplitude with depth is clearly observed.

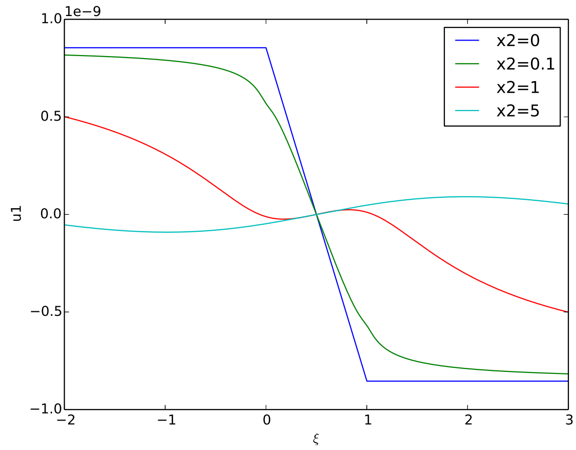


Figure 4: Horizontal displacement for a distributed load.

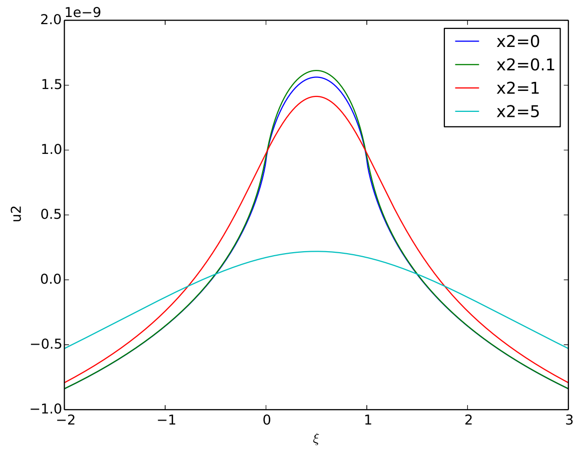


Figure 5: Vertical displacement for a distributed load.

## 3 Rigid punch

The same methodology may be applied to mixed boundary value problems [15]. In particular, consider the steady-state problem for a rigid punch moving along the boundary of an elastic half-plane $\mathcal{H}_2^+$ at a constant speed $c < c_2$ in the absence of friction, see Fig. 6.

The equations of motion in the moving coordinate frame $(\xi, x_2)$ are again taken in the form (8), whereas now the boundary conditions along the surface $x_2 = 0$ are

$$\begin{aligned} \sigma_{22} = 0, \quad \xi \in S_1; \\ u_3 = f(\xi), \quad \xi \in S_2; \end{aligned} \tag{25}$$

and

$$\sigma_{\xi 3} = 0, \quad -\infty < \xi < \infty, \tag{26}$$

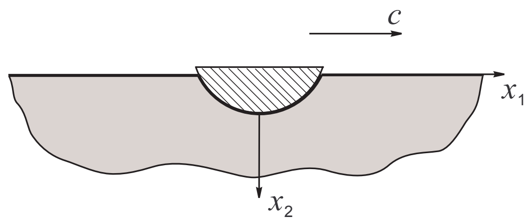


Figure 6: Moving punch.

where $S_1 \cup S_2 = \mathbb{R}$. Due to the last condition (26), potential $\psi$ is a Hilbert tranform of $\phi$, with (12) holding true. On introducing the auxiliary function

$$\phi_1 = \frac{\alpha_2^2 - 1}{\alpha_2^2 + 1}\,\phi_{,2} \tag{27}$$

and scaling $z = \alpha_1 x_2$, it is possible to reduce the first equation (8) with (25) to a conventional mixed boundary value problem for the Laplace equation. Thus, we have

$$\phi_{1,zz} + \phi_{1,\xi\xi} = 0 \tag{28}$$

subject to

$$\begin{aligned} \phi_1 = f(\xi), \quad \xi \in S_2; \\ \phi_{1,\xi} = 0, \quad \xi \in S_1, \end{aligned} \tag{29}$$

along the surface $z = 0$.

## 4 Concluding remarks

The presented approach allows a straightforward treatment of the subsonic regime of a moving load on an elastic half-space. In particular, the original vector problem in elasticity is reduced to a scalar problem for the Laplace equation.

The described approach has a potential to be generalised to the three-dimensional setup, see [16], and also to layered [17] and anisotropic media, see e. g. [18] and [19]. Another direction of extension is associated with moving disturbances along the interface of two media, see [16] and [20]. There is also a possibility of simplified analysis of a moving load on a beam resting on an elastic half-space, e. g. see [21].

## References

[1] Cole, J., Huth, J., 1958, Stresses produced in a half plane by moving loads, *ASME J. Appl. Mech.*, Vol. **25**, pp. 433–436.

[2] Gakenheimer, D. C., Miklowitz, J., 1969, Transient excitation of an elastic half space by a point load traveling on the surface, *ASME J. Appl. Mech.*, Vol. **36**, pp. 505–515.

[3] Georgiadis, H. G., Barber, J. R., 1993, Steady-state transonic motion of a line load over an elastic half-space: The corrected Cole–Huth solution, *ASME J. Appl. Mech.*, Vol. **60**, pp. 772–774.

[4] Kaplunov, J. D., 1986, *Transient dynamics of an elastic half-plane subject to a moving load.* Preprint No. 277, Institute for Problems in Mechanics, Moscow, (in Russian).

[5] Kaplunov, J., Nolde, E., Prikazchikov, D. A., 2010, A revisit to the moving load problem using an asymptotic model for the Rayleigh wave, *Wave motion*, Vol. **47**, pp. 440–451.

[6] Kaplunov, J., Zakharov, A., Prikazchikov, D. A., 2006, Explicit models for elastic and piezoelastic surface waves, *IMA J. Appl. Math.*, Vol. **71**, pp. 768–782.

[7] Kaplunov, J., Prikazchikov, D. A., 2013, Explicit models for surface, interfacial and edge waves, In: *Dynamic Localization Phenomena in Elasticity, Acoustics and Electromagnetism*, Eds.: R. V. Craster, J. Kaplunov, pp. 73–114, Springer-Verlag, Vienna.

[8] Kaplunov, J., Prikazchikov, D. A., Erbas, B., Sahin, O., 2013, On a 3D moving load problem in an elastic half space, *Wave Motion*, Vol. **50**, pp. 1229–1238.

[9] Erbaş, B., Kaplunov, J., Prikazchikov, D. A., Şahin, O., 2014, The near-resonant regimes of a moving load in a three-dimensional problem for a coated elastic half-space, *Math. Mech. Solids*, doi: 10.1177/1081286514555451.

[10] Chadwick, P. 1976, Surface and interfacial waves of arbitrary form in isotropic elastic media, *J. Elast.*, Vol. **6**, pp. 73–80.

[11] Friedlander, F. G., 1948, On the total reflection of plane waves, *Q. J. Mech. Appl. Math.*, Vol. **1**, pp. 376–384.

[12] Sobolev S. L., 1937, Some problems in wave propagation. In: *Differential and Integral Equations of Mathematical Physics*, Eds.: P. Frank, R. von Mises, ONTI, Moscow–Leningrad, pp. 468–617, (in Russian).

[13] Achenbach, J. D., 1973, *Wave Propagation in Elastic Solids*, North-Holland, Amsterdam.

[14] Courant R., Hilbert D., 1989, *Methods of Mathematical Physics. Vol. 2*, John Wiley & Sons, New York.

[15] Erbaş, B., Kaplunov, J., Prikazchikov, D. A., 2013, The Rayleigh wave field in mixed problems for a half-plane, *IMA J. Appl. Math.*, Vol. **78**, pp. 1078–1086.

[16] Kiselev, A. P., Parker, D. F., 2010, Omni-directional Rayleigh, Stoneley and Schölte waves with general time dependence, *Proc. Roy. Soc. A*, Vol. **466**, pp. 2241–2258.

[17] Dai, H.-H., Kaplunov, J., Prikazchikov, D. A., 2010, A long-wave model for the surface elastic wave in a coated half-space, *Proc. Roy. Soc. London A*, Vol. **466**, pp. 3097–3116.

[18] Prikazchikov, D. A., 2013, Rayleigh waves of arbitrary profile in anisotropic media, *Mech. Res. Comm.*, Vol. **50**, pp. 83–86.

[19] Parker, D. F., 2013, The Stroh formalism for elastic surface waves of general profile, *Proc. Roy. Soc. A*, Vol. **469**, doi: 10.1098/rspa.2013.0301.

[20] Parker, D. F., 2012, Evanescent Schölte waves of arbitrary profile and direction, *Eur. J. Appl. Math.*, Vol. **23**, pp. 267–287.

[21] Fryba, L., 1972, *Vibration of Solids and Structures Under Moving Loads*, Noordhoff International, Groningen, The Netherlands.