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Transient near tip fields in crack dynamics†

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Transient effects of stress-strain fields in the vicinity of a stationary crack tip under high rate loads are discussed. Exact analytical solutions to near tip stresses are compared to fields prescribed by leading terms (one or several) of Williams asymptotic expansion. Influence of load application mode, time (or, which is the same, distance from a crack tip) and Poisson's ratio on this discrepancy is extensively examined. Some effects connected with crack tip propagation speed are also discussed. Significant inconsistencies between real (or received in numerical solutions of state equations – e.g. finite element computations) crack tip fields and stress intensity factor (SIF) singular field observed by numerous researchers are explained. The scope of problems where SIF field can be used for correct prediction of dynamic stress-strain fields in the crack tip region is established. Possibility to correctly approximate fields that are not SIF dominated, accounting additional terms of Williams expansion, is studied.

transient crack tip fields, dynamic fracture, high-rate loads, asymptotic expansions

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List of main symbols

σ_{ij}	components of the Cauchy stress tensor
t	time
r,θ	polar coordinates with origin at the crack tip
K_I	the first term of the asymptotic expansion of stresses surrounding the tip of the mode I loaded crack (the mode I stress intensity factor)
R_n	the second and the following terms of the asymptotic expansion of stresses surrounding the crack tip
ϕ_{ij}	angular functions
$x(x_1, x_2)$	Cartesian coordinate
$W=W(t,x)$	displacement field
u,v	components of displacement
μ	shear modulus
ν	Poisson's ratio
E	Young's modulus
$c_1=\sqrt{\dfrac{1-\nu}{(1+\nu)(1-2\nu)}\dfrac{E}{\rho}}=\sqrt{\dfrac{2(1-\nu)}{1-2\nu}\dfrac{\mu}{\rho}}$	longitudinal wave speed
$c_2=\sqrt{\dfrac{\mu}{\rho}}$	transversal wave speed
c_R	Rayleigh wave speed
$f(t)$	time dependent load
P	load amplitude
H	Heaviside step function equal to 0 if the argument is negative and equal to 1 otherwise
φ, ψ	longitudinal and transversal wave potentials
ρ	mass density
$\gamma=c_2/c_1=\sqrt{\dfrac{1-2\nu}{2(1-\nu)}}$	ratio of longitudinal and transverse wave speeds
$\gamma_R=c_R/c_1$	ratio of Rayleigh and longitudinal wave speeds

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S_i the sum of the first i terms of the asymptotic expansion

σ_c the critical value for tensile stress (ultimate strength)

K_{1C} the critical value for stress intensity factor

τ the microstructural time of a brittle fracture process (or fracture incubation time) – a parameter characterizing the response of the studied material to applied dynamic loads. τ is constant for a given material and does not depend on problem geometry, the way a load is applied, the shape of a load pulse and its amplitude

d characteristic size of fracture process zone

1 Introduction

Transient effects connected with dynamic effects of impact loaded cracks have been studied and observed analytically, numerically and experimentally for the last 50 years. Remarkable analytical solutions in crack dynamics belong to Yoffe [1], Ang [2,3], Freund [4], Achenbach [5–7], Eshelby [8], Broberg [9,10] and Kostrov [11–13]. As a particular case, Kostrov's solution gives an exact representation for stress-strain fields in the vicinity of an impact loaded stationary semi-infinite crack. A similar solution will be extensively used in this paper as a reference result, which will be compared to stress-strain fields prescribed by leading terms of the Williams asymptotic expansion [14]. Though utilizing Kostrov's approach one can achieve solutions to a wide range of problems and loads (it raises a possibility to construct a solution for arbitrary moving cracks subjected to arbitrary loads), the result is normally very complicated and hard or even impossible to analyze. This is one of the reasons why the stress intensity factor is traditionally used to describe stressed conditions surrounding a crack tip.

Another reason for this is that the first approaches in fracture dynamics were connected with attempts to migrate Irwin's approach [15], successful for the majority of materials, geometries and loads in static conditions, directly into a dynamical situation.

Williams expansion [14] of crack tip stress field for mode I loaded crack reads:

$$\sigma_{ij}(t,r,\theta)=\frac{K_{\mathrm{I}}(t)}{\sqrt{2\pi r}}\cdot\phi_{ij}(1,\theta)+\sum_{n=2}^{\infty}R_n(t)r^{\frac{n}{2}-1}\cdot\phi_{ij}(n,\theta), \quad (1)$$

where σ stands for stress depending on time t, distance to the crack tip r and angle θ, indices i and j assume the values 1 and 2, and K_{I} is the mode I stress intensity factor, changing with time. Angular functions $\phi_{ij}(n,\theta)$ are given by:

$$\phi_{11}(n,\theta)=\left(2+\frac{n}{2}+(-1)^n\right)\times\cos\left[\left(\frac{n}{2}-1\right)\theta\right]-\left(\frac{n}{2}-1\right)\cos\left[\left(\frac{n}{2}-3\right)\theta\right],$$

$$\phi_{22}(n,\theta)=\left(2-\frac{n}{2}-(-1)^n\right)\times\cos\left[\left(\frac{n}{2}-1\right)\theta\right]-\left(\frac{n}{2}-1\right)\cos\left[\left(\frac{n}{2}-3\right)\theta\right],$$

$$\phi_{12}(n,\theta)=\phi_{12}(n,\theta)=\left(\frac{n}{2}-1\right)\sin\left[\left(\frac{n}{2}-3\right)\theta\right]-\left(\frac{n}{2}+(-1)^n\right)\sin\left[\left(\frac{n}{2}-1\right)\theta\right].$$

In static conditions $K_{\mathrm{I}}(t)$ and $R_n(t)$ are constants. While solving quasistatic problems, the first singular term of Williams expansion normally gives a good representation of stress field adjacent to a crack tip. In this case analysis of critical fracture conditions can be done utilizing only stress intensity factor (SIF)-Irwin's critical SIF criterion is applicable. In the dynamic case $K_{\mathrm{I}}(t)$ and $R_n(t)$ change with time. Each of these functions will depend not only on time but on loads applied as well. Therefore accuracy of singular K_{I} field will depend not only on a point location (i.e. r and θ) as in statics but even on time.

Numerous researchers observed that K_{I} field does not always correctly reflect results it receives while numerically solving dynamic problems of linear fracture mechanics (e.g. utilizing the finite element method (FEM), the boundary element method (BEM) or meshless methods) [16]. They observed that for some class of problems, dynamic field surrounding the vicinity of a crack tip is not K_{I} dominated.

Though Kostrov's solution is known for more than 50 years and is applicable in a wide variety of problems, there is no general unanimity among researchers working in fracture dynamics field about conditions and reasons leading to appearance of these transient effects. Some authors correctly associate this with impossibility to apply the K_{I} singular field while describing some of the extremely dynamic problems.

This paper is an attempt to determine the range of problems for which the singular field created in a crack tip region is not K_{I} dominated. For such problems possibility to represent stress-strain fields with a finite number of terms of Williams expansion is also studied. It will also be shown that for some problems Williams expansion is not converging to real stress field. In this case only the exact solution can give a correct description of a dynamic process.

2 Problem formulation and analytical solution 1: anti-plane case

Infinite elastic plane with a semi-infinite cut $\{(x_1, x_2)\}$: $x_2=\pm 0$, $x_1\leqslant 0$ is considered. Displacement field is given by

$W = W(t, x)$, where t stands for time and x is coupled with stresses by

$$\sigma_{x_1x_3} = \mu\frac{\partial W}{\partial x_1},\quad \sigma_{x_2x_3} = \mu\frac{\partial W}{\partial x_2}. \tag{2}$$

where μ is the shear modulus and W satisfies the wave equation:

$$\frac{\partial^2 W}{\partial x_1^2} + \frac{\partial^2 W}{\partial x_2^2} = \frac{1}{c_2^2}\frac{\partial^2 W}{\partial t^2}, \tag{3}$$

with c_2 being the speed of the transversal wave. For negative times media are stress free:

$$W\big|_{t<0} = 0. \tag{4}$$

On the cut $\{(x_1, x_2)\}$: $x_2=\pm 0$, $x_1\leqslant 0$, we suppose:

$$\sigma_{x_2x_3}\Big|_{\substack{x_2=\pm 0\\ x_1<0}} = 0. \tag{5}$$

To receive unique solutions of eqs. (2)–(5) requires absence of energy sources in the vicinity of a crack tip:

$$W = O(r^{\lambda}),\quad r = \sqrt{x_1^2 + x_2^2} \to 0,\quad \lambda > 0,\quad \forall t \geqslant \delta > 0. \tag{6}$$

Solutions of eqs. (2)–(6) are well-known (ex. [17]). For the case of $f(t) = -PH(t)$, where $H(t)$ is the Heaviside step function, solution for stresses on crack continuation gives:

$$\sigma_{x_2x_3} = \begin{cases} 0, & c_2t < x_1, \\ \dfrac{2\mathrm{P}}{\pi}\left(\sqrt{\dfrac{c_2t}{x_1} - 1} - \arctan\sqrt{\dfrac{c_2t}{x_1} - 1}\right), & c_2t \geqslant x_1. \end{cases} \tag{7}$$

Expanding eq. (7) into series one can get:

$$\sigma_{x_2x_3} = P\left(\frac{2\sqrt{c_2t}}{\pi\sqrt{x_1}} - 1 + \frac{\sqrt{x_1}}{\pi\sqrt{c_2t}}\right) + \mathrm{O}\left[\left(\frac{x_1}{c_2t}\right)^{\frac{3}{2}}\right],\quad x_1 \to 0,\ x_1 \leqslant c_2t. \tag{8}$$

Corresponding value of the stress intensity factor in this case will be:

$$K_{\mathrm{I}}(t) = \frac{2\sqrt{2}P}{\sqrt{\pi}}\sqrt{c_2t}. \tag{9}$$

Suppose that the impact is not applied on the crack faces, but is delivered to the crack region by a wave generated by a load applied at infinity:

$$\sigma_{x_2x_3}\left(t, x_1, x_2\right)\Big|_{t<0} = f\left(t + \frac{x_2}{c_2}\right). \tag{10}$$

In this case eq. (5) is substituted by:

$$\sigma_{x_2x_3}\Big|_{\substack{x_2=\pm 0\\ x_1<0}} = 0. \tag{11}$$

If $f(t) = P\cdot H(t)$, then the solutions of eqs. (2), (3), (6), (10) and (11) give:

$$\sigma_{x_2x_3} = \begin{cases} P, & c_2t < x_1, \\ \dfrac{2P}{\pi}\left(\sqrt{\dfrac{c_2t}{x_1} - 1} - \arctan\sqrt{\dfrac{c_2t}{x_1} - 1}\right) + P, & c_2t \geqslant x_1, \end{cases} \tag{12}$$

for stresses on continuation of the crack. Stress intensity factor time dependence will be the same as eq. (9) and series expansion of $\sigma_{x_2x_3}$ will differ from eq. (8) by eliminated constant pressure term $-P$:

$$\sigma_{x_2x_3} = P\left(\frac{2\sqrt{c_2t}}{\pi\sqrt{x_1}} + \frac{\sqrt{x_1}}{\pi\sqrt{c_2t}}\right) + \mathrm{O}\left[\left(\frac{x_1}{c_2t}\right)^{\frac{3}{2}}\right],\quad x_1 \to 0,\ x_1 \leqslant c_2t. \tag{13}$$

Using eqs. (7) and (12) it is easy to construct a solution for arbitrary $f(t)$. Corresponding result is achieved using time convolution of eq. (7) or eq. (12) with $f(t)$. Arbitrary load can be presented as a convolution with the Heaviside step function:

$$f(t) = \int_{-\infty}^{\infty} H(s) f'(t - s)\mathrm{d}s. \tag{14}$$

Then

$$\sigma^{f}_{x_2x_3}(t, x_1, x_2) = \int_{-\infty}^{\infty} \sigma_{x_2x_3}(s, x_1, x_2)\ f'(t - s)\mathrm{d}s,$$

where $\sigma_{x_2x_3}(s, x_1, x_2)$ is taken from eq. (7) for the case of load applied on the crack faces or eq. (12), for the case of load delivered by a wave, will give a solution to stresses.

Later the solution for load that is linearly growing with time will be used. In this case

$$f(t) = V\,t\,H(t) = V\int_{-\infty}^{\infty} H(s)H(t - s)\mathrm{d}s.$$

For the case of load applied at crack faces the solution to stresses on crack continuation reads:

$$\sigma_{x_2x_3} = \begin{cases} 0, & c_2t < x_1, \\ \dfrac{2V}{\pi c_2}\left(\dfrac{2}{3}\dfrac{(c_2t - x_1)^{3/2}}{\sqrt{x_1}} + \sqrt{x_1(c_2t - x_1)} - c_2t\arctan\sqrt{\dfrac{c_2t}{x_1} - 1}\right), & c_2t \geqslant x_1. \end{cases} \tag{15}$$

Series expansion of eq. (15) gives:

$$\sigma_{x_2x_3} = Vt\left(\frac{4\sqrt{c_2 t}}{3\pi\sqrt{x_1}} - 1 + \frac{2\sqrt{x_1}}{\pi\sqrt{c_2 t}}\right) + \mathrm{O}\left[\left(\frac{x_1}{c_2 t}\right)^{\frac{3}{2}}\right], \quad (16)$$

$$x_1 \to 0, \quad x_1 \leqslant c_2 t.$$

Corresponding stress intensity factor will be:

$$K_1(t) = \frac{4\sqrt{2}V}{3\sqrt{\pi}} t\sqrt{c_2 t}. \quad (17)$$

3 Problem formulation and analytical solution 2: plane case

The plane dynamic problem of elasticity is considered. Homogeneous isotropic infinite plane has a semi-infinite cut $\{(x_1, x_2)\}$: $x_2=\pm 0$, $x_1 \leqslant 0$. Stress field is given by potentials φ and ψ, satisfying the following conditions:

$$\frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2} = \frac{1}{c_1^2}\frac{\partial^2\varphi}{\partial t^2},$$
$$\frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} = \frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2}. \quad (18)$$

Here φ and ψ are longitudinal and transversal wave potentials, and c_1 is the speed of longitudinal wave. Components of displacement u and v are coupled with φ and ψ by:

$$u = \frac{\partial\varphi}{\partial x_1} + \frac{\partial\psi}{\partial x_2},$$
$$v = \frac{\partial\varphi}{\partial x_2} - \frac{\partial\psi}{\partial x_1}. \quad (19)$$

Crack faces are free from tractions:

$$\sigma_{x_2x_2}\Big|_{\substack{x_2=\pm 0\\ x_1\leqslant 0}} = 0,$$
$$\sigma_{x_1x_2}\Big|_{\substack{x_2=\pm 0\\ x_1\leqslant 0}} = 0. \quad (20)$$

Initial conditions are given by a wave approaching the crack region from infinity:

$$\psi\big|_{t<0} = 0,$$
$$\varphi\big|_{t<0} = H\left(t + \frac{x_2}{c_1}\right). \quad (21)$$

It is requested that displacements are bounded at an area adjacent to the crack tip, which guarantees the uniqueness of the solution.

Stresses can be evaluated via potentials:

$$\sigma_{x_1x_1} = \rho c_1^2\left[\frac{\partial^2\varphi}{\partial x_1^2} + \left(1-2\gamma^2\right)\frac{\partial^2\varphi}{\partial x_2^2} + 2\gamma^2\frac{\partial^2\psi}{\partial x_1\partial x_2}\right],$$

$$\sigma_{x_2x_2} = \rho c_1^2\left[\frac{\partial^2\varphi}{\partial x_2^2} + \left(1-2\gamma^2\right)\frac{\partial^2\varphi}{\partial x_1^2} - 2\gamma^2\frac{\partial^2\psi}{\partial x_1\partial x_2}\right], \quad (22)$$

$$\sigma_{x_1x_2} = \rho\gamma^2 c_1^2\left[2\frac{\partial^2\varphi}{\partial x_1\partial x_2} - \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2}\right],$$

where ρ is the mass density and $\gamma = c_2/c_1$.

The solution [18] to eqs. (18)–(22), received for wave potentials is:

$$\Psi'(\varOmega) = \frac{\mathrm{i}\sqrt{2}\gamma_R G(\varOmega)\sqrt{1+\varOmega}}{\pi\gamma^3\sqrt{1-\gamma^2}\left(1+\gamma_R\varOmega\right)},$$
$$\varPhi'(\varOmega) = \frac{\mathrm{i}\gamma_R G(\varOmega)\left(1-2\gamma^2\varOmega^2\right)}{\pi\sqrt{2}\,\gamma\,\sqrt{1-\gamma^2}\,\varOmega\,\left(1+\gamma_R\varOmega\right)\sqrt{1-\varOmega}}, \quad (23)$$

where $\psi = \mathrm{Re}(\Psi)$ and $\varphi = \mathrm{Re}(\varPhi)$, with Ψ and $\varPhi$ being analytical everywhere. $\varOmega$ is a coordinate on a complex plane. $\gamma_R = c_R/c_1$, where c_R is the Rayleigh wave speed.

$$G(\varOmega) = \exp\left(\frac{1}{\pi}\int_{-1/\gamma}^{-1}\frac{\arg\left(R(s)\right)}{\varOmega - s}\mathrm{d}s\right),$$

with $R(z)$ being the Rayleigh function:

$$R(z) = \left(1-2\gamma^2 z^2\right) + 4\gamma^3 z^2\sqrt{1-z^2}\sqrt{1-\gamma^2 z^2}$$

and $\arg(z)$ being the argument function:

$$\arg(z) = -\mathrm{i}\log\frac{z}{|z|}.$$

By substituting eq. (23) into eq. (19), displacements can be found. Presented formulas give the solution for load created by elementary longitudinal wave. Solution for load created by elementary transversal wave can be achieved analogously. Solutions for more complex loads can be achieved as a time convolution of presented solution or/and solution for transversal wave.

Using the presented solution and eq. (22) stresses on continuation of a semi-infinite crack for the problem when a load is delivered to the crack region by a falling wave, Stresses on continuation can be found:

$$\sigma_{x_2x_2} = \rho\frac{c_1^2}{r^2}\mathrm{Re}\left[\left(1-2\gamma^2\left(\frac{c_1 t}{r}\right)^2\right)\varPhi''\Big|_{x_2=0} - 4\gamma^2\frac{c_1 t}{r}\varPhi'\Big|_{x_2=0}\right.$$
$$\left. + 2\mathrm{i}\gamma\left(\frac{c_1 t}{r}\sqrt{\gamma^2\left(\frac{c_1 t}{r}\right)^2 - 1}\,\Psi''\Big|_{x_2=0} + \frac{2\gamma^2\left(\frac{c_1 t}{r}\right)^2 - 1}{\sqrt{\gamma^2\left(\frac{c_1 t}{r}\right)^2 - 1}}\Psi'\Big|_{x_2=0}\right)\right]. \quad (24)$$

Load corresponding to constant pressure suddenly applied on the crack faces can be presented as a wave:

$$\psi\big|_{t<0} = 0,$$
$$\varphi\big|_{t<0} = \frac{P}{2\rho}\left(t+\frac{x_2}{c_1}\right)^2 H\left(t+\frac{x_2}{c_1}\right). \quad (25)$$

Convolution of eq. (24) with loads given by eq. (21) or eq. (25) will give exact solutions to the plane problem with load delivered to the crack region by a falling wave or a load applied directly on the crack faces.

Evaluating asymptotic expansion for stresses on continuation of a cut in this problem one will get:

$$\sigma_{x_2x_2} = P\frac{2\sqrt{2}\sqrt{c_1}\gamma\sqrt{1-\gamma^2}}{\pi}\left[\frac{\sqrt{t}}{\sqrt{x_1}} - \frac{(\gamma_R+2\gamma_R R_1-2)}{2c_1\gamma_R}\frac{\sqrt{x_1}}{\sqrt{t}}\right] - P + \mathrm{O}\left[\left(\frac{x_1}{c_1 t}\right)^{3/2}\right], \quad (26)$$

for the case of impact applied on the crack faces and

$$\sigma_{x_2x_2} = P\frac{2\sqrt{2}\sqrt{c_1}\gamma\sqrt{1-\gamma^2}}{\pi}\left[\frac{\sqrt{t}}{\sqrt{x_1}} - \frac{(\gamma_R+2\gamma_R R_1-2)}{2c_1\gamma_R}\frac{\sqrt{x_1}}{\sqrt{t}}\right] + \mathrm{O}\left[\left(\frac{x_1}{c_1 t}\right)^{3/2}\right], \quad (27)$$

when the load is delivered to the crack by a wave approaching from infinity.

4 Accuracy of asymptotic representation of stress state surrounding dynamically loaded crack tip

At this section accuracy of presented asymptotic solutions will be analyzed. For the comparison of stresses evaluated accounting several first terms of Williams expansion to exact analytical solution the following expression is introduced:

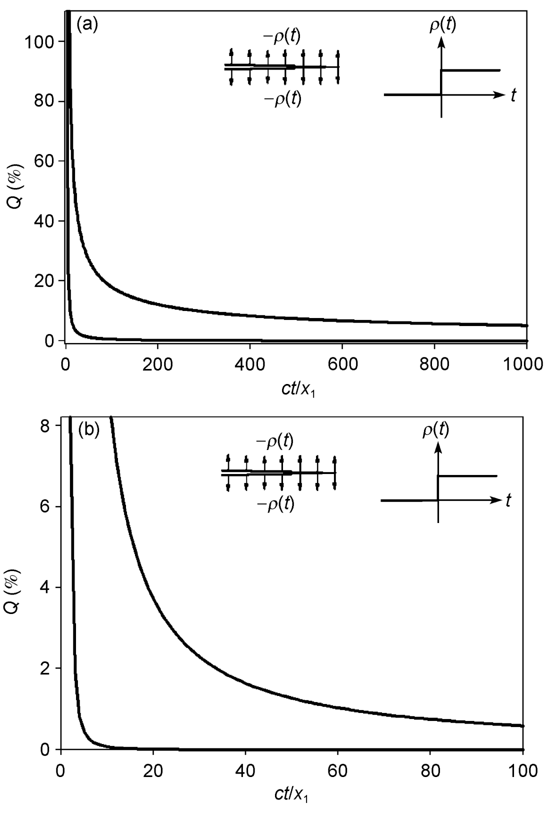
$$Q = \left[\frac{\sigma - S_i}{\sigma}\right]\cdot 100\%. \quad (28)$$

Here $\sigma = \sigma_{x_2x_3}$ for the anti-plane case and $\sigma = \sigma_{x_2x_2}$ for the plane problem. S_i is the sum of the first i terms of Williams expansion (1). Thus, eq. (28) gives a relative error of asymptotic approximation.

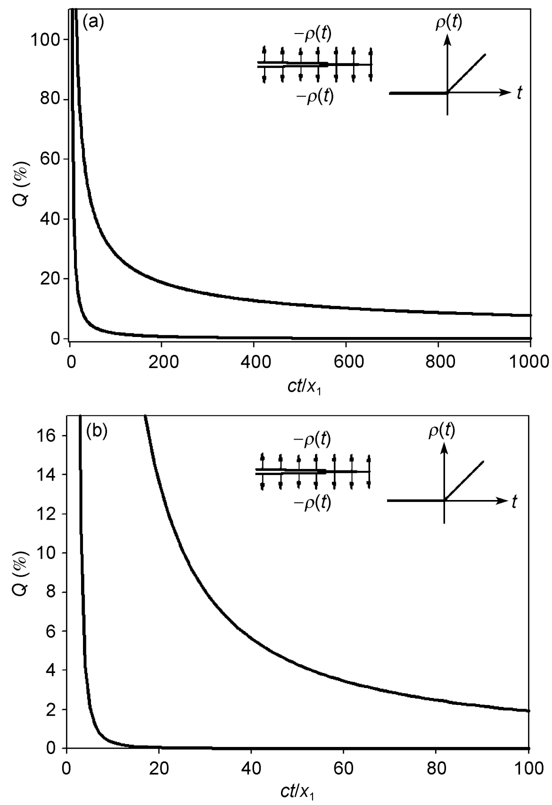
To start with, behavior of Q at the anti-plane problem with load suddenly applied on the crack faces is discussed. To evaluate Q in this situation one should use stress σ given by eq. (7) and S_i given by eq. (8), taking the first 1, 2 or 3 terms. Results are presented in Figures 1(a) and 1(b).

The horizontal axis in Figures (Figures 1(a), 1(b) and all the following figures) stands for dimensionless value ct/x_1. This value shows the distance from the studied point x to the position on the crack continuation where the wave front is currently situated. After the front has passed point x, $ct/x_1>1$ is fulfilled. According to computational results given in Figure 1(a), if the front of the wave is less than $100x_1$ away from the point with coordinate x_1 on the crack continuation, representation using only singular term of the Williams expansion – stress intensity factor (upper curve in Figure 1(a)) is appreciably incorrect (by more than 20%). For $ct/x_1>200$ the misfit is considerably reduced (less than 10%). When the second term of Williams expansion is taken into consideration (lower curve in Figure 1(a) or upper curve in Figure 1(b)), the situation is improved significantly. In this case already for $ct/x_1>10$ error is less than 10%. Taking the third term of expansion into account (lower curve in Figure 1(b)) improves the result even more. Already for $ct/x_1>2$ the error is below 10%. For $ct/x_1>5$ misfit between the result received using the first three terms of Williams expansion and the exact solution is less than 1%.

To begin with, a problem for crack faces loaded by uniformly distributed pressure growing in time with a constant rate V is studied. To receive the desired error estimation in this case one should substitute corresponding exact solution eq. (15) and approximation eq. (16) into eq. (28). The respective curves are presented in Figures 2(a) and 2(b). As it



Figures 1 Relative error of asymptotic series solution of the anti-plane problem with constant load suddenly applied on the crack faces. (a) Upper curve-first term (SIF), lower curve-two first terms; (b) upper curve-two terms, lower curve-three terms.



Figures 2 Relative error of asymptotic series solution of the anti-plane problem with load growing at a constant rate suddenly applied on the crack faces. (a) Upper curve-first term (SIF), lower curve-two first terms; (b) upper curve-two terms, lower curve-three terms.

is seen from these figures, behavior of the misfit between the exact solution and the first terms of asymptotic expansion remains qualitatively unchanged. The accuracy is slightly reduced.

The next problem to be analyzed is the problem with stress free crack faces and a load given by a wave with constant amplitude P, moving from infinity with a front parallel to the crack. The exact expression for stresses in this problem is given by eq. (12) and an approximate asymptotic solution differs from eq. (8) by absence of $-P$ term. Substituting these solutions into eq. (28) and performing computations one can receive data presented in Figure 3. As one can see in this problem the accuracy of approximation using only stress intensity factor is essentially better. It is even more exact than representation using two first terms of Williams expansion in the previous problems (Figures 1(a) and 2(a)). This is connected with the fact that in this case the term following stress intensity factor $K(t)$ in eq. (1) is missing ($R_0(t)=0$).

The analogues analysis is performed for solution of the problem in the plane case. The first of the examined load options is a uniformly distributed pressure suddenly applied on the crack faces ($\sigma_{x_2x_2}=-PH(t)$, $\sigma_{x_{12}x_2}=0$). The solution for $\sigma_{x_2x_2}$ on the crack continuation is given by the

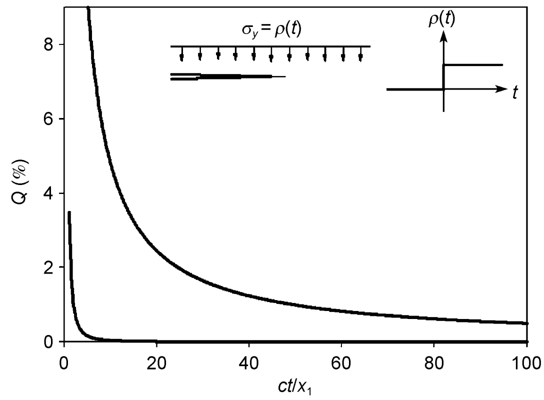


Figure 3 Relative error of asymptotic series solution of the anti-plane problem with a load given by a wave with constant amplitude, moving from infinity with a front parallel to the crack. Upper curve-first term (SIF), lower curve-two first terms.

convolution of stress given by eq. (24) with load eq. (25). The corresponding asymptotic solution is given by eq. (27). In this case both the exact and the asymptotic solutions are depending on the ratio between longitudinal and transversal wave speeds $\gamma=c_2/c_1$. Misfit between the solution achieved using only singular term of the asymptotic expansion (K) and exact solution is presented in Figure 4 for different values of γ. The approximation in a limiting case of Poisson's ratio ν equal to 0 ($\gamma=1/\sqrt{2}$) is not strictly accurate as compared with the corresponding anti-plane problem results. Decrease of γ (increase of ν) results in reduction of accuracy of approximation using the stress intensity factor. This is explained by the fact that the multiplier at square root singular term in the anti-plane problem ($1/\pi\approx0.64$) is bigger than the multiplier at the plane case ($2\sqrt{2}\gamma\sqrt{1-\gamma^2}\pi$, with $\gamma=1/\sqrt{2}-0.45$). Figures 5(a) and 5(b) give a comparison of accuracy of the asymptotic approximation using one (stress intensity factor), two and three first terms (while

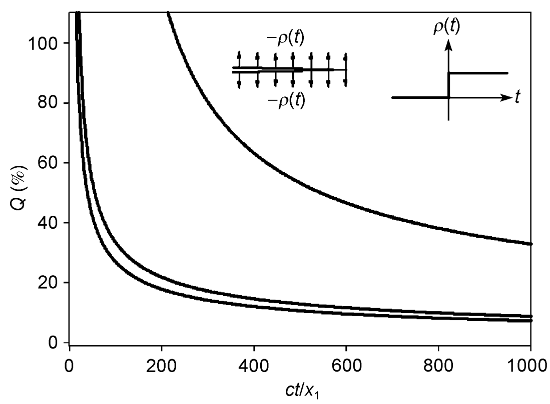
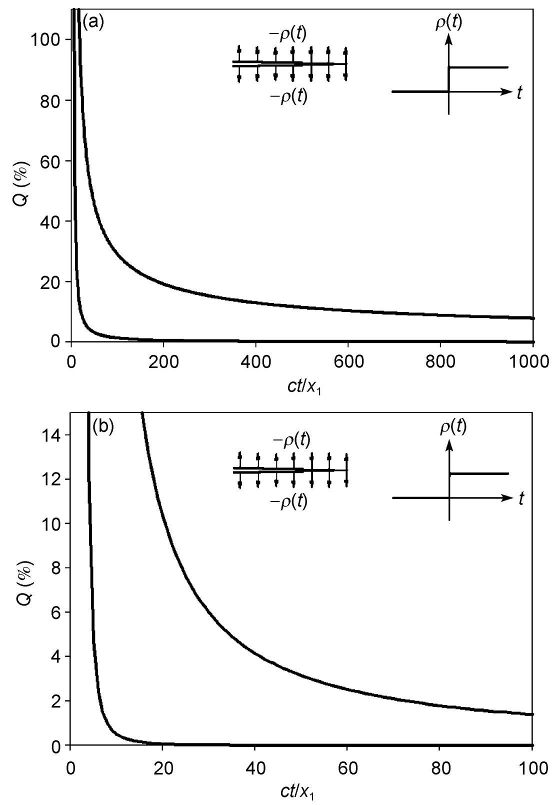


Figure 4 Relative error of SIF solution of the plane problem with constant load suddenly applied on the crack faces. Upper curve: γ=0.14 (ν =0.49), middle curve: γ=0.48 (ν =0.35), lower curve: $\gamma=1/\sqrt{2}$ (ν =0).



Figures 5 Relative error of asymptotic series solution of the plane problem with constant load suddenly applied on the crack faces. (a) Upper curve-first term (SIF), lower curve-two first terms; (b) upper curve-two terms, lower curve-three terms.

$\gamma=1/\sqrt{3}$ and $\nu=0.25$). Character of the curves is close to those presented in Figure 1(a). As already discussed above, the accuracy of asymptotic approximations is lower in the plane case.

For example, at a point on crack continuation, distant from the tip of the crack by 10 mm and for a material with longitudinal wave speed c_1=5000 m/s, error received using the stress intensity factor approximation will exceed 20% for times t<400 microseconds. Only for times exceeding 1200 microseconds the error is below 10%.

Figure 6 presents the misfit between approximation using stress intensity factor or two first terms of asymptotic expansion eq. (26) and the exact solution of the plane problem for a crack loaded by a wave, approaching from infinity with a front parallel to the crack (convolution of eq. (24) with eq. (21)). Stress distribution inside the wave is given by Heaviside step function. The situation is similar to the anti-plane problem (Figure 3).

As demonstrated above in the case of load applied on the crack faces (Figures 1(a), 1(b) and 5(a), 5(b)), the approximation using the first term of the Williams expansion is less accurate as compared with the analogues problem where the load is created by a passing wave (Figures 3 and 6). This is due to the absence of regular terms (terms non-depending

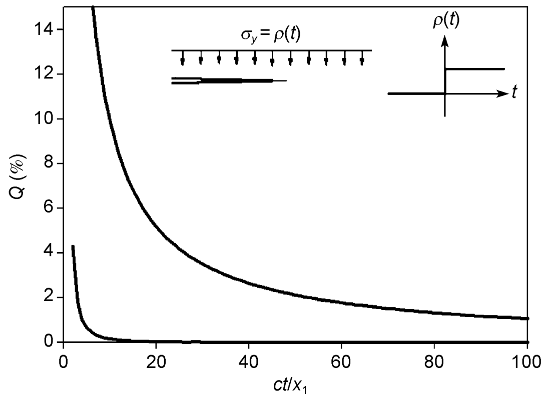


Figure 6 Relative error of asymptotic series solution of the plane problem with a load given by a wave with constant amplitude, moving from infinity with a front parallel to the crack. Upper curve-first term (SIF), lower curve-two first terms.

on the coordinate) in the asymptotic expansion of solutions in the case of load created by a falling wave.

It can be demonstrated that in the case of the load applied on the crack faces the same effect can be achieved as a result of a special choice of a time shape for the load function. In order to do this rectangular shaped load pulse with amplitude P and duration T ($f(t)=P[H(t)-H(t-T)]$) is applied on the crack faces. Anti-plane conditions are assumed. In this situation for times $t>T$ term independent of the coordinate (regular term) is vanishing. Figure 7 plots the misfit between approximation given by the stress intensity factor and the exact solution for the problem. One can see that for $t>T$ the accuracy of solution given by stress intensity factor is noticeably increased.

5 Influence of transient stress field on fracture in the tip of the crack

In this section we provide the analysis of influence that the

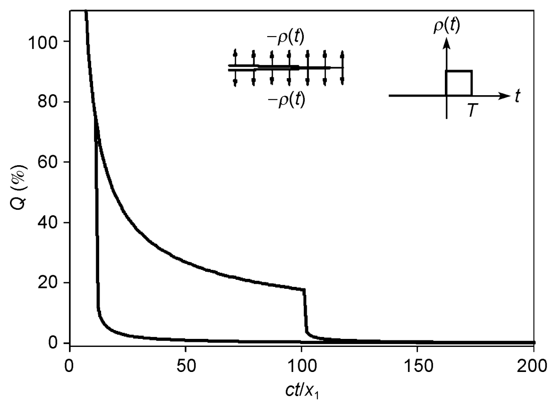


Figure 7 Relative error of SIF solution of the anti-plane problem with constant load of duration T suddenly applied on the crack faces. Upper curve, $T=100x_1/c$; lower curve, $T=10\,x_1/c$.

above contribution of terms following the stress intensity factor term in asymptotic expansion into stress field can have on critical load amplitudes leading to fracture in the tip of the loaded crack.

In refs. [19–21] criterion for fracture applicable in the studied situation of highly dynamic loading conditions was proposed

$$\frac{1}{\tau}\int_{t-\tau}^{t}\frac{1}{d}\int_{0}^{d}\sigma(x,t)\mathrm{d}x\mathrm{d}t \leqslant \sigma_c, \tag{29}$$

where τ stands for incubation time of fracture [21], being material property independent of experimental conditions, σ_c stands for the ultimate stress of the studied material, evaluated in quasistatic conditions, and d stands for characteristic size [21], having a dimension of length and given by $d=(2/\pi)\cdot(K_{Ic}^2/\sigma_c^2)$, where K_{Ic} is the critical value for stress intensity factor evaluated in quasistatic conditions and $\sigma(x,t)$ is giving normal stress an point x ant time t. x=0 in this situation corresponds to the crack tip. Exhaustive information about ideology and physics behind the incubation time fracture criterion and the incubation time can be found in the book by Morozov and Petrov [21]. In the same book one can find possible experimental schemes that can be used in order to evaluate the incubation time. Eq. (29) provides a possibility to predict moment of time t^* when fracture is initiated in the tip of the crack. The same analysis can predict critical amplitude for a load applied to the crack.

Notion of minimal fracturing load pulse will be used below. Under minimum fracturing load pulse we will mean load pulse of a given duration with minimum possible amplitude resulting in initiation of the studied crack. For any load duration it is possible to find corresponding minimum amplitude leading to fracture initiation. In some cases fracture will happen with a certain delay (counting from the moment of load termination to fracture initiation). This delay cannot exceed fracture incubation time τ. For short pulses (shorter then τ) the delay increases if the load duration is decreased. Increasing the load amplitude while its duration is preserved will result in decrease of the fracture delay. Load pulse of a given duration that results in initiation of fracture without delay (simultaneously with load termination) will be called maximum fracturing load pulse. Further increase of the load amplitude while its duration is preserved will lead to fracture initiation before the load is terminated.

In order to analyze effect of accuracy of the asymptotic representation on the critical load characteristics leading to initiation of fracture in the tip of the crack the following expression is introduced:

$$Q_i=\left|\frac{P_i-P_3}{P_3}\right|\cdot 100\% \tag{30}$$

P_i is the critical (minimal) amplitude of rectangular shaped load pulse initiating fracture in the tip of the loaded crack calculated utilizing eq. (29) by using first i terms of the asymptotic expansion.

The solutions presented above do not account for the wave cone created at the crack tip: when time t is comparable to time needed for the wave to travel a distance equal to structural size d, asymptotic expansion can be only used for $t>x/c$. For $t<x/c$ the wave that is diffracted from the tip of the crack has not reached the point x. Thereby, stress in this point for $t<x/c$ is given by the stress in the falling wave. Thus, accounting for the wave cone, created at the tip of the crack will improve the solution.

Calculations of accuracy of predictions of critical load amplitudes for different load durations using different number terms of the asymptotic expansion are performed for real materials. We will use material properties typical for PMMA: $c_1=1970$ M/C, $c_2=1130$ M/C, $\sigma_c=20$ MPa, $K_{Ic}=1.47$ MPa m$^{1/2}$, $\tau=30$ μs. Results of calculations are presented in Figure 8.

Figure 8 gives Q_1 (upper curves) and Q_2 (lower curves) for rectangular threshold (with minimum fracturing amplitude) load pulse as a function of load duration T. Solid line gives a solution received taking the wave cone into account. Dashed line solution is received not taking this into account. The received results (see Figure 8) indicate that taking into account only the first term of the Williams asymptotic expansion will result in up to 30% error in estimation of amplitude for minimum fracturing load pulse. Using the first two terms of the expansion drops maximum error below 10%.

Figure 9 gives Q_1 (upper curves) and Q_2 (lower curves) for rectangular maximum fracturing load pulse as a function of load duration T (or, which is the same in this case, time-to-fracture t_*). Solid line gives a solution received taking the wave cone into account. Dashed line solution is received not taking this into account. The received results (see

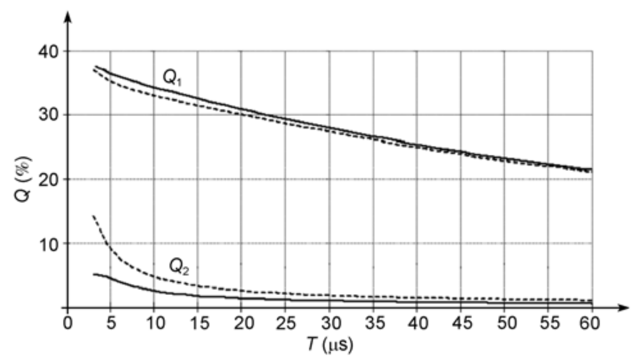


Figure 8 Accuracy of the computed minimum fracturing load pulse amplitude for rectangular load pulse as a function of load pulse duration. Upper curves-first term of the asymptotic expansion is used. Lower curves-first and second terms are used.

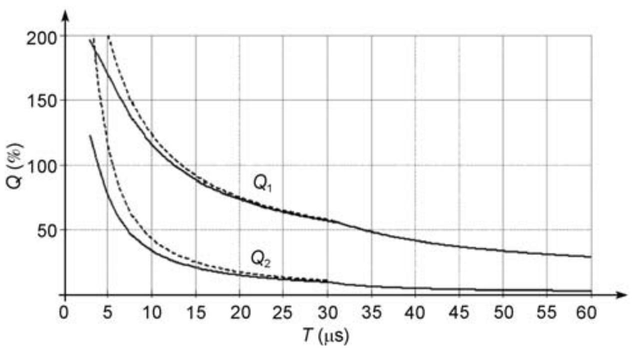


Figure 9 Accuracy of the computed maximum fracturing load pulse amplitude for rectangular load pulse as a function of load pulse duration. Upper curves-first term of the asymptotic expansion is used. Lower curves-first and second terms are used.

Figure 9) indicate that taking into account only the first term of the Williams asymptotic expansion will result in up to 200% error in estimation of amplitude for maximum fracturing load pulse. Using the first two terms of the expansion significantly reduces the error.

6 Conclusions

As clearly demonstrated by previous examples, behavior of asymptotic representations of stress-strain fields in a vicinity of a crack tip in dynamic problems is characterized by substantial non-uniformity. Obviously the accuracy of representation using just stress intensity factor in dynamic problems cannot be sufficiently increased by introduction of special "dynamic correction", even time dependent. To achieve a correct solution in dynamic conditions one should account terms of the Williams asymptotic expansion following the SIF term.

Very interesting observations through examining dependency of accuracy SIF stress field approximation can be made in dynamic problems on Poisson's ratio of studied material (Figure 4). One can note that for incompressible materials (ν=1/2) infinite number of terms in Williams asymptotic expansion should be taken in order to obtain reasonable coincidence between approximation and reality. It is demonstrated (Figure 4) that the larger the Poisson's ratio of material, the worse the approximation using the SIF. For a material with ν=0.48 the misfit between real stress and stress prescribed by SIF square root singularity exceeds 20% even for times c_1t/x_1=1000.

Presented results demonstrate the framework for problems where the SIF can be used to describe stress-strain fields surrounding the tip of a dynamically loaded crack. It is shown how a load (both the way a load is applied and its time shape), material properties (Poisson's ratio) and experimental conditions (plane or anti-plane problem) can affect accuracy of asymptotic approximations for stress-strain field in a vicinity of a crack tip. It is shown that in many cases, when the SIF square root singular field can not provide a correct approximation of dynamic stress field surrounding the crack tip, accounting one or two additional terms following the SIF in power expansion can greatly improve the situation. At the same time there are situations when an infinite number of terms is needed in order to approximate solutions in an accurate way.

Although presented solutions refer to stationary dynamically loaded cracks, we also want to discuss applicability of SIF approximation of singular stress field surrounding the crack tip in problems with propagating cracks. As discussed by Morozov and Petrov [21], the closer the crack tip speed is approaching the Rayleigh wave speed C_R, the worse the approximation given by the SIF for stress field in a vicinity of the tip of the crack is. Therefore, for moving cracks speed is another important factor that is affecting the SIF approximation accuracy. Having this in mind one may wish to revise Freund's solution for the limiting speed of crack propagation [22] for mode I cracks and solutions for permitted speeds for shear cracks [4,22–25]. Though we are unlikely to question the fact that C_R is the limiting speed for mode I cracks in problems without local scale and microstructure, as there are other physical and empirical reasons why mode I cracks cannot propagate with greater speeds, obviously, revision of solutions for shear crack propagation can help better understand recent experiments on ultrasonic dynamic cracking [26].

Hopefully demonstrated exact solutions and comparisons of exact solutions to SIF prescribed singular stress fields will help understand the mismatch between SIF solutions and more exact solutions, observed by multiple researchers.

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