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MOVING LOAD ON AN ELASTIC HALF-SPACE COATED WITH A THIN VERTICALLY INHOMOGENEOUS LAYER

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Abstract. *The study is focussed on the near-critical regimes of the moving load on a coated elastic half-space. The material properties of the coating are assumed to be depending on the vertical coordinate. The analysis relies on the hyperbolic-elliptic formulation for the Rayleigh wave induced by prescribed surface loading, containing an elliptic equation for the elastic potential governing its decay over the interior, and a singularly perturbed wave equation on the interface between the layer and the substrate. This scalar formulation allows significant simplifications, including in particular, classification of the near-resonant regimes. The method of fictitious absorption is then used in order to incorporate the effect of poles, associated with the radiation of energy from the moving source. Finally, numerical illustrations of results for several types of vertical inhomogeneity are presented.*

1 INTRODUCTION

Moving load problems on elastic structures have been studied for around a century now, see e.g. [12], [25] and possess important industrial applications, in particular, in high-speed train operation [4]. Solution for steady-state regime of a moving load on an elastic half-space has been first presented in [6], see also [14]. Moving load on a coated half-space has been seemingly first considered by [2]. More realistic modelling motivates development of studies oriented to dynamics of multi-layered and vertically inhomogeneous half-space, see e.g. [15] and [22].

At the same time, sophisticated structure of the exact solution stimulates development of effective approximations. In view of the well-known fact of the Rayleigh wave speed being the critical speed of the moving load, the near-resonant formulation for surface wave field, allowing explicit solutions seems a prospective approach. The current work is based on the long-wave asymptotic formulation presented in [7], which has been applied to problems on a moving load for elastic half-space [17, 10, 8, 26], see also [18] for a more systematic exposition of the methodology of hyperbolic-elliptic models for surface waves. Recent advances of the approach include in particular extensions incorporating the effects of anisotropy [13] and pre-stress [21], composite models for elastic layers [9], as well as development of seismic meta-surfaces [29] and the second-order refined model [30].

The described asymptotic model [18] is focused on the contribution of the Rayleigh wave to the overall dynamic response. It is derived as a slow-time near-resonant perturbation of the self-similar solution by [5] and hence allows reduction of the vector problem of elasticity to a scalar problem for the Laplace equation in respect of the longitudinal elastic potential, with the boundary condition on the surface presented in the form of the forced wave equation. In case of the coated half-space the presence of the layer is reflected by the appropriate pseudo-differential operator, thus, the boundary condition describing the near-surface dynamics is a singularly perturbed wave equation, see e.g. [7].

The focus of the current contribution is on the near-resonant steady-state regimes of a moving line load on a half-space coated by a vertically inhomogeneous thin coating. The corresponding extension of the hyperbolic-elliptic model has been recently suggested in [24]. The study of a singularly perturbed hyperbolic equation at the interface allows classification of the regimes of the moving load, pointing out typical behaviours depending on the speed of the load and the combinations of material parameters.

The paper is organized as follows. The problem is formulated in Section 2. The analytic treatment of the pseudo-differential equation on the interface between the layer and the substrate and classification of regimes of the moving load is carried out in Section 3. Finally, numerical illustrations of the results are presented in Section 4.

2 FORMULATION OF THE PROBLEM

Consider a linearly isotropic, elastic half-space over the domain $-\infty < x_1, x_2 < \infty$ and $x_3 \geq 0$, coated by a thin layer of thickness h described by $-\infty < x_1, x_2 < \infty$ and $-h \leq x_3 \leq 0$, see Fig. 1.

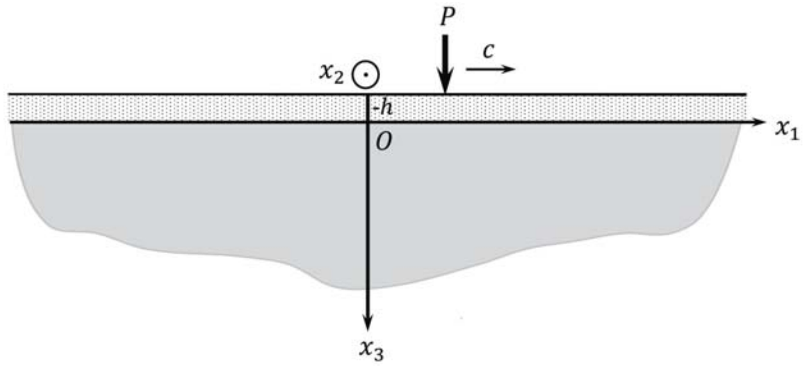


Figure 1: Moving load on a coated elastic half-space.

The properties of the coating are assumed to be dependent on the transverse variable x_3, so that

$$\lambda_c = \lambda(x_3), \quad \mu_c = \mu(x_3), \quad \rho_c = \rho(x_3) \tag{1}$$

denote the Lamé elastic parameters and volume mass density, respectively. The substrate is assumed homogeneous, with the associated material parameters given by λ_s, μ_s and ρ_s. The governing equations of motion and the constitutive relations of linear isotropic elasticity are conventionally written as

$$\sigma^q_{ij,j} = \rho_q u^q_{i,tt}, \qquad \sigma^q_{ij} = \lambda_q u^q_{k,k} + \mu_q \left(u^q_{i,j} + u^q_{j,i}\right), \tag{2}$$

where σ^q_{ij} and u^q_i $(i, j = 1, 2, 3)$ are the Cauchy stress and displacement components of the coating and substrate $(q = c, s)$, comma indicates differentiation with respect to appropriate spatial or time variable, and the Einstein summation convention is adopted. Here we restrict the consideration to plane-strain formulation, for which $u^q_2 = 0$ and the components $u^q_1 = u^q_1(x_1, x_3, t)$ and $u^q_3 = u^q_3(x_1, x_3, t)$ are independent of x_2.

The loading on the surface of the coating is prescribed in the form of a line vertical force P, moving at a constant speed c, hence the boundary conditions are written as

$$\sigma^c_{31} = 0, \qquad \sigma^c_{33} = P_0\, \delta\left(x_1 - ct\right), \qquad \text{at} \quad x_3 = -h, \tag{3}$$

see Fig.1. Perfect bonding on the interface is assumed, giving $(m = 1, 3)$

$$u^s_m = u^c_m, \qquad \sigma^s_{3m} = \sigma^c_{3m}, \qquad \text{at} \quad x_3 = 0. \tag{4}$$

Below, the long-wave assumption is adopted, i.e.

$$\epsilon = \frac{h}{L} \ll 1, \tag{5}$$

where L is the typical wave length.

We consider the steady-state problems focusing on the near-resonant regimes, when the speed of the moving load is close to that of the resonant Rayleigh wave c_R in the substrate, namely

$$\left| \frac{c}{c_R} - 1 \right| \ll 1. \tag{6}$$

Then, the contribution of the Rayleigh wave to the overall dynamic response is dominant compared to that of body waves, hence, the hyperbolic-elliptic model for the Rayleigh wave is applicable.

The corresponding asymptotic formulation for a vertically inhomogeneous coated half-space has been recently proposed in [24], extending the previous results in [7]. Within this framework, efficient boundary conditions are derived by means of the asymptotic integration technique, see e.g. [11] and [19], then serving as a starting point for slow-time perturbation procedure.

For the current problem, the proposed model for surface wave contains an elliptic equation for the interior $x_3 > 0$

$$\frac{\partial^2 \phi}{\partial x_3^2} + \alpha_R^2 \frac{\partial^2 \phi}{\partial x_1^2} = 0, \tag{7}$$

where ϕ is the longitudinal Lamé elastic potential in the substrate and

$$\alpha_R = \sqrt{1 - \frac{\rho_s c_R^2}{\lambda_s + 2\mu_s}}. \tag{8}$$

The boundary condition at $x_3 = 0$ is given by a singularly perturbed 1D hyperbolic equation

$$\frac{\partial^2 \phi}{\partial x_1^2} - \frac{1}{c_R^2}\frac{\partial^2 \phi}{\partial t^2} - bh\sqrt{-\frac{\partial^2}{\partial x_1^2}}\left(\frac{\partial^2 \phi}{\partial x_1^2}\right) = -\frac{1+\beta_R^2}{2\mu B} P_0\, \delta\left(x_1 - ct\right), \tag{9}$$

where

$$B = \frac{\alpha_R}{\beta_R}\left(1 - \beta_R^2\right) + \frac{\beta_R}{\alpha_R}\left(1 - \alpha_R^2\right) - 1 + \beta_R^4, \qquad \beta_R = \sqrt{1 - \frac{\rho_s c_R^2}{\mu_s}}, \tag{10}$$

and the constant b is given by

$$b = \frac{\left(1 - \beta_R^2\right)}{2\,\mu\, B}\left[\tilde{\rho}\, c_R^2\, \left(\alpha_R + \beta_R\right) - \beta_R\, \tilde{\gamma}\right], \tag{11}$$

with the integral quantities accounting for vertical inhomogeneity

$$\tilde{\gamma} = \frac{4}{h} \int_{-h}^{0} \frac{\mu_c (\lambda_c + \mu_c)}{\lambda_c + 2\mu_c} \, dx_3, \qquad \text{and} \qquad \tilde{\rho} = \frac{1}{h} \int_{-h}^{0} \rho_c \, dx_3. \tag{12}$$

It is worth noting that the pseudo-differential operator $\sqrt{-\frac{\partial^2}{\partial x_1^2}}$ is an essential feature of the problem, associated with the dispersion of waves within the coating layer. Moreover, the sign of the coefficient b is associated with the sign of the group velocity in the long-wave limit, see [28] for more detail.

Once the longitudinal potential ϕ is determined from the scalar formulation (7), (9), the transverse potential ψ may be found as its harmonic conjugate, see e.g. [18], originating from [5]

$$\psi (x_1, \beta_R x_3, t) = \frac{2\alpha_R}{1 + \beta_R^2} \, \phi^* (x_1, \beta_R x_3, t) \, , \tag{13}$$

where the asterisk denotes the Hilbert transform.

Now, the boundary value problem (7), (9) may be rewritten in the moving coordinate system $(\xi, x_3) = (x_1 - ct, x_3)$, hence the steady-state limit is governed by

$$\frac{\partial^2 \phi}{\partial x_3^2} + \alpha_R^2 \, \frac{\partial^2 \phi}{\partial \xi^2} = 0, \tag{14}$$

subject to

$$\eta \, \frac{\partial^2 \phi}{\partial \xi^2} - b \, h \sqrt{-\frac{\partial^2}{\partial \xi^2}} \left(\frac{\partial^2 \phi}{\partial \xi^2} \right) = - \frac{1 + \beta_R^2}{2\mu B} \, P_0 \, \delta(\xi) \qquad \text{at} \qquad x_3 = 0, \tag{15}$$

where

$$\eta = 1 - \frac{c^2}{c_R^2}.$$

3 ANALYSIS ON THE INTERFACE

Let us concentrate on the analysis of equation (15) on the interface $x_3 = 0$. Note that on setting $h = 0$ the problem formulation (14), (15) will reduce to that for an uncoated elastic half-space which is the leading order Taylor expansion of the exact solution [6], for more details see [18].

Another observation which immediately follows from (15) is the presence of two small parameters, a geometric one, associated with the long-wave approximation, as well as η corresponding to the near-resonant vicinity.

Now, let us introduce the dimensionless coordinate $\zeta = \left|\frac{\eta}{b}\right| \frac{\xi}{h}$ along with a scaled quantity

$$\chi = -\frac{2\mu\, B\, b\, h}{(1+\beta_R^2)\, P_0} \frac{\partial^2 \phi}{\partial \xi^2}. \tag{16}$$

Then, the equation (15) takes the form

$$\operatorname{sgn}(b\,\eta)\ \chi - \sqrt{-\frac{\partial^2}{\partial \zeta^2}}\,(\chi) = \ \delta(\zeta), \tag{17}$$

The solution of the latter can be obtained by using the Fourier integral transform

$$\chi = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{e^{i\omega\zeta}\, d\omega}{\operatorname{sgn}(b\,\eta) - |\omega|}. \tag{18}$$

From (18) it may be observed that the analysis splits in two sub-cases, depending on the value of sgn $(b\,\eta)$.

3.1 SUB-CASE 1: NO POLES ON THE REAL AXE

Consider first the situation $b\eta < 0$, in which

$$\chi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega\zeta}\, d\omega}{-1 - |\omega|} = \mathcal{F}\left(|\zeta|\right), \tag{19}$$

where

$$\mathcal{F}\,(x) = \frac{1}{\pi}\left[\text{si}(x)\sin x + \text{Ci}(x)\cos x\right], \tag{20}$$

with si and Ci denoting the sine and cosine integral functions, respectively, i.e.

$$\text{si}(x) = -\int\limits_{x}^{\infty} \frac{\sin t}{t}\, dt, \qquad \text{Ci}(x) = -\int\limits_{x}^{\infty} \frac{\cos t}{t}\, dt,$$

see e.g. [1]. Note that this case occurs either in the the sub-Rayleigh regime $(c < c_R)$ with local minimum of the phase velocity at the Rayleigh wave speed $(b < 0)$, or in the super-critical regime $(c > c_R)$ combined with local maximum of the phase velocity $(b > 0)$, for more detail see [7].

3.2 SUB-CASE 2: POLES ON THE REAL AXE

The second sub-case $b\eta > 0$ of poles on the real axis of the integral (18) may be treated using the limiting absorption principle, see e.g. [27, 16].

In case of the sub-critical regime $(c < c_R)$, when $(b > 0)$, formula (18) yields

$$\chi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega\zeta}\, d\omega}{1 - |\omega|} = -2\, H\left(-\zeta\right) \sin\zeta + \mathcal{F}\left(|\zeta|\right), \tag{21}$$

whereas in the case of the super-Rayleigh regime $(c > c_R)$ the solution becomes

$$\chi = 2\, H\left(\zeta\right) \sin\zeta + \mathcal{F}\left(|\zeta|\right), \tag{22}$$

with $\mathcal{F}$ defined in (20) and $H(\zeta)$ denoting the Heaviside function.

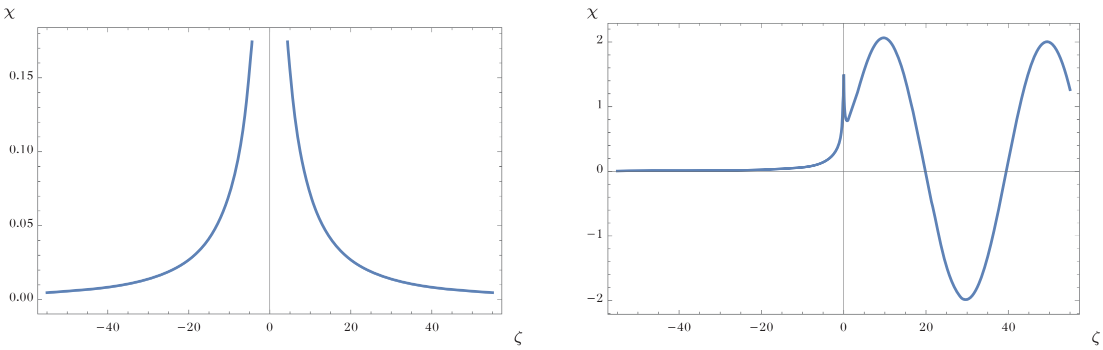
4 NUMERICAL RESULTS

In this section, the illustrations of the obtained results are presented. We consider the dependence of the Young's modulus E_c of the coating layer on the vertical coordinate x_3 of exponential form, see e. g. [22]

$$E(x_3) = E_c\, e^{\beta\, x_3}, \;\; \beta = \frac{1}{h} \ln\left(\frac{E_s}{E_c}\right), \tag{23}$$

providing a smooth variation from the value E_c on the surface of the coating to the Young's modulus E_s of the homogeneous substrate. In the computations below we also take $h = 1$ and assume constant mass densities $\rho_c = \rho_s = 1$, and Poisson's ratios $\nu_c = \nu_s = 0.25$, with the sub-Rayleigh and super-Rayleigh regimes computed for $c = 0.9\, c_R$, and $c = 1.1\, c_R$, respectively.

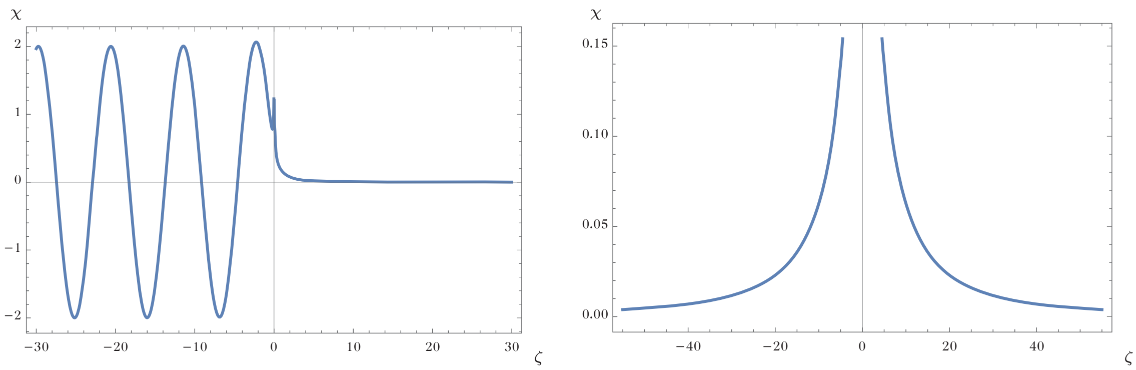
Consider first the case of a relatively hard coating layer, when $E_c/E_s = 10$. The associated graphs in Fig. 2 show the dependence of the quantity χ on the scaled moving coordinate ζ. In this case, the constant b may be computed using (11), resulting in $b \approx -1.32$. In Fig. 2(a) the super-Rayleigh regime is depicted $(\eta > 0)$, corresponding to the case of no poles in (18), whereas Fig. 2(b) illustrates the sub-Rayleigh regime $(\eta < 0)$, clearly showing the effect of poles in (18) for positive ζ, i.e. radiation of energy in front of the moving source.



(a) $\eta > 0$. (b) $\eta < 0$.

Figure 2: Dependence of the quantity χ on the moving coordinate ζ for the case of softening within the layer ($E_c/E_s = 10$).

The following Figs. 3 (a) and (b) illustrate another typical scenario, when the coating layer is relatively soft compared to the substrate and stiffens gradually with depth, when $E_c/E_s = 0.1$. The calculation of the constant b according to (11) gives $b \approx 0.276$. The plots in Fig. 3(a) and 3(b) show the variation of the quantity χ on the moving coordinate ζ. Now, the energy radiation behind the moving load is observed in the super-Rayleigh regime in Fig. 3 (a).



(a) $\eta > 0$. (b) $\eta < 0$.

Figure 3: Dependence of the quantity χ on the moving coordinate ζ for the case of hardening within the layer ($E_c/E_s = 0.1$).

5 CONCLUSIONS

The steady-state problem for a moving line load on a half-space coated by a thin vertically inhomogeneous layer has been analysed. The hyperbolic-elliptic formulation for the Rayleigh wave field allowed an explicit solution, as well as a clear classification of the regimes.

Further possible developments include addressing transient problems [17], generalizations to coatings with more sophisticated mechanical properties, taking into account the effects of viscosity, curvature, long-wave high-frequency waves [20], as well as a more general treatment of a vertically inhomogeneous half-space, see e.g. [3]. We also mention potential applications for seismic metabarriers [23]

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REFERENCES

[1] M. Abramowitz, I. A. Stegun. Handbook of mathematical functions. New York, NY: Dover, 1982.

[2] J. D. Achenbach, S. P. Keshava, G. Herrmann. Moving load on a plate resting on an elastic half space. Journal of Applied Mechanics, 34, 910–914, 1967.

[3] I. Argatov, I., A. Iantchenko. Rayleigh surface waves in functionally graded materials – long-wave limit. Quarterly Journal of Mechanics and Applied Mathematics 72(2) 197–211 (2019).

[4] Y. Cao, H. Xia, Z. Li. A semi-analytical/FEM model for predicting ground vibrations induced by high-speed train through continuous girder bridge. Journal of Mechanical Science and Technology, 26(8), 2485–2496, 2012.

[5] P. Chadwick. Surface and interfacial waves of arbitrary form in isotropic elastic media. Journal of Elasticity, 6, 73–80, 1976.

[6] J. Cole, J. Huth. Stresses produced in a half plane by moving loads. Journal of Applied Mechanics, 25, 433–436, 1958.

[7] H.-H. Dai, J. Kaplunov, D. A. Prikazchikov. A long wave asymptotic model for surface wave in a coated elastic half-space.Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 466, 3097–3116, 2010.

[8] N. Ege, B. Erbaş. Response of a 3D elastic half-space to a distributed moving load. Hacettepe Journal of Mathematics and Statistics, 46(5), 817–828, 2017.

[9] B. Erbaş, J., Kaplunov, E., Nolde, M. Palsü. Composite wave models for elastic plates. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 474(2214), 20180103, 2018.

[10] B. Erbaş, J. Kaplunov, D. A. Prikazchikov, O. Şahin. The near-resonant regimes of a moving load in a three-dimensional problem for a coated elastic half-space. Mathematics and Mechanics of Solids, 22(1), 89-100, 2017.

[11] B. Erbaş, B., E. Yusufoglu, J. Kaplunov. A plane contact problem for an elastic orthotropic strip. Journal of Engineering Mathematics, 70(4), 399–409, 2011.

[12] L. Fryba. Vibration of solids and structures under moving loads. London: Thomas Telford, 1999.

[13] Y. Fu, J. Kaplunov, D. A. Prikazchikov. Reduced model for the surface dynamics of a generally anisotropic elastic half-space. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 476(2234), 20190590, 2020.

[14] H. G. Georgiadis, J. R. Barber. Steady-state transonic motion of a line load over an elastic half-space. The corrected Cole/Huth solution. ASME Transactions Series E Journal of Applied Mechanics, 60, 772–774, 1993.

[15] E. V. Glushkov, N. V. Glushkova, S. I. Fomenko, C. Zhang. Surface waves in materials with functionally gradient coatings. Acoustical Physics, 58(3), 339–353, 2012.

[16] V. V. Kalinchuk, T. I. Belyankova. Dynamics of the surface of inhomogeneous media. Moscow: Fizmatlit, 2009.

[17] J. Kaplunov, E. Nolde, D. A. Prikazchikov. A revisit to the moving load problem using an asymptotic model for the Rayleigh wave. Wave Motion, 47, 440–451, 2010.

[18] J. Kaplunov, D. A. Prikazchikov. Asymptotic theory for Rayleigh and Rayleigh-type waves. Advances in Applied Mechanics, 50, 1–106, 2017.

[19] J. Kaplunov, D. A. Prikazchikov, L. Sultanova. Justification and refinement of Winkler–Fuss hypothesis. Zeitschrift für Angewandte Mathematik und Physik, 69(3), 80, 2018.

[20] J. Kaplunov, D. A. Prikazchikov, L. Sultanova. Rayleigh-type waves on a coated elastic half-space with a clamped surface. Philosophical Transactions of the Royal Society A, 377(2156), 20190111, 2019.

[21] L. A. Khajiyeva, D. A. Prikazchikov, L. A. Prikazchikova. Hyperbolic-elliptic model for surface wave in a pre-stressed incompressible elastic half-space. Mechanics Research Communications, 92, 49–53, 2018

[22] R. Kulchytsky-Zhyhailo, A. S. Bajkowski. Three-dimensional analytical elasticity solution for loaded functionally graded coated half-space. Mechanics Research Communications, 65, 43–50, 2015.

[23] S. V. Kuznetsov. Seismic waves and seismic barriers. Acoustical Physics, 57(3), 420–426, 2011.

[24] A. Mubaraki, D. Prikazchikov. Explicit model for surface waves on an elastic half-Space coated by a thin vertically inhomogeneous layer. Proceedings of the 9th International Conference Dynamical Systems-Theory and Applications DSTA, 2019.

[25] G. B. Muravskii. Mechanics of non-homogeneous and anisotropic foundations. Springer, Berlin, Heidelberg, 2012.

[26] O. Şahin. Analysis of the Rayleigh wave field due to a tangential load applied on the surface of a coated elastic half-space. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 69(1), 158–171, 2020.

[27] J. R. Schulenberger, C. H. Wilcox. The limiting absorption principle and spectral theory for steady-state wave propagation in inhomogeneous anisotropic media. Archive for Rational Mechanics and Analysis, 41, 46–65, 1971.

[28] A. L. Shuvalov, A. G. Every. On the long-wave onset of dispersion of the surface-wave velocity in coated solids. Wave Motion, 45, 857–863, 2008.

[29] P. T. Wootton, J. Kaplunov, D. J. Colquitt. An asymptotic hyperbolic–elliptic model for flexural-seismic metasurfaces. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 475(2227), 20190079, 2019.

[30] P. T. Wootton, J. Kaplunov, D. A. Prikazchikov. A second-order asymptotic model for Rayleigh waves on a linearly elastic half-plane. IMA Journal of Applied Mathematics, 85, 113–131, 2020.