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# Seismic barriers filled with granular metamaterials: Mathematical models for granular metamaterials

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**Abstract.** The problem of seismic protection from the main types of surface acoustic waves and shear–pressure (SP) evanescent waves emanating from vicinity of an epicenter of an earthquake is discussed. Herein, SP waves represent a kind of the evanescent waves arising at critical angles of incident of bulk shear waves. The proposed seismic protection method utilizes vertical trenches (vertical barriers) filled with the specially constructed granular metamaterials. Some of nonlinear hyperelastic models along with nonlinear and inelastic models are analyzed for applications using granular metamaterials in case of cyclic dynamic loadings that correspond to arrival of the large intensity surface acoustic and evanescent waves. The main attention is paid to arrival of the large intensity Rayleigh, Rayleigh–Lamb and SP waves, as the most frequent waves and the most dangerous waves for engineering structures. Some of the new constitutive equations for metamaterials exhibiting different elastic moduli at tension and compression phases are proposed and discussed.

## 1. Introduction

Granular metamaterials used as filler for seismic barriers and seismic cushions are extensively studied in various laboratories around the world [1–25]. For analyzing mechanical properties of the discussed granular metamaterials at cyclic dynamic loadings, several methods are proposed, including elastic, hyperelastic and hypoelastic equations of state [26–29], elastic-plastic [1–3, 6, 7, 30], viscoelastic-plastic [19, 20], hydrodynamic equations of state [21, 22], etc. Herein, various nonlinear hyperelastic equations of state are analyzed.

The problem of seismic protection from the main types of surface acoustic waves and shear–pressure (SP) evanescent waves emanating from vicinity of the epicenter of an earthquake is discussed. The proposed seismic protection method utilized vertical trenches (vertical barriers) filled in with the specially constructed granular metamaterials. Herein, the following notation is applied, the term vertical barrier in contrast to the horizontal barrier, is referred to a vertical formation in the upper layers of the Earth crust. The typical vertical dimension for a vertical barrier is chosen as a quarter of the wavelength that should be reduced in the area behind the barrier [31–33]. In practice, depth of the typical vertical barrier may vary from 4 to 25 m. Another remark concerns the typical dimensions of the artificial pebbles used as fillings for the barriers, according to [33, 34] the recommended diameters lie in a range of 0.01–0.1 m, however, for some seismic pads used in bridges, the diameters could be much larger, approaching 0.5–1 m, see [35]. Some of the nonlinear hyperelastic models along with nonlinear and inelastic models are analyzed for applications of the usage of granular metamaterials in case of cyclic dynamic loadings that correspond to arrival of the large intensity surface acoustic and evanescent waves. The main attention is paid to arrival of the large intensity Rayleigh, Rayleigh–Lamb and SP waves, as the most frequent waves and most dangerous waves for the engineering structures. Some of the new constitutive equations for metamaterials exhibiting different elastic moduli at tension and compression phases are proposed and discussed.

## 2. Hyperelastic potentials

### *2.1. Equations of state*

At the assumption of the infinitesimally small deformations, the stress–strain relation for the hyperelastic material takes the form

$$\boldsymbol{\sigma} = \lambda(\mathrm{I}_\varepsilon, \mathrm{II}_\varepsilon, \mathrm{III}_\varepsilon)\mathrm{I}_\varepsilon \boldsymbol{I} + 2\mu(\mathrm{I}_\varepsilon, \mathrm{II}_\varepsilon, \mathrm{III}_\varepsilon)\varepsilon, \tag{1}$$

where $\boldsymbol{I}$ denotes the unit diagonal matrix; Lame's $\lambda$ and $\mu$ are functions of the strain (or stress) invariants, that can be written in the following form

$$\mathrm{I}_\varepsilon \equiv \mathrm{tr}(\boldsymbol{\varepsilon}), \qquad \mathrm{II}_\varepsilon \equiv \tfrac{1}{2}\left(I_{\boldsymbol{\varepsilon}}^2 - \boldsymbol{\varepsilon}\cdot\cdot\varepsilon\right), \qquad \mathrm{III}_\varepsilon \equiv \det(\boldsymbol{\varepsilon}). \tag{2}$$

Note that equation (1) ensures existence of the scalar potential, that will be introduced below in (6). It will be assumed that the strain energy that relates to equation (1), ensures that the condition of strong ellipticity of the corresponding elastic tensor is satisfied [26,30]. The assumed strong ellipticity condition requires the following inequalities: $\mu > 0$, $3\lambda + 2\mu > 0$.

### *2.2. Equations of motion for the nonlinear media*

Introducing Cauchy relation for the infinitesimal strain tensor and the displacement field [24]

$$\boldsymbol{\varepsilon} = \tfrac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^t\right) \tag{3}$$

and substituting equation of state (1) into (infinitesimal) equation of motion, yields

$$(\lambda + 2\mu)\,\nabla_x \mathrm{div}_x \boldsymbol{u} - \mu\,\mathrm{rot}_x\,\mathrm{rot}_x\,\boldsymbol{u} + (\nabla_x \lambda)\,\mathrm{div}_x \boldsymbol{u} + \nabla_x \mu\left(\nabla_x \boldsymbol{u} + \nabla_x \boldsymbol{u}^t\right) = \rho \ddot{\boldsymbol{u}}, \tag{4}$$

where in view of (1)

$$\nabla_x \lambda = \left(\frac{\partial \lambda}{\partial \mathrm{I}_\varepsilon}\nabla_x \mathrm{I}_\varepsilon + \frac{\partial \lambda}{\partial \mathrm{II}_\varepsilon}\nabla_x \mathrm{II}_\varepsilon + \frac{\partial \lambda}{\partial \mathrm{III}_\varepsilon}\nabla_x \mathrm{III}_\varepsilon\right). \tag{5}$$

The gradient $\nabla_x \mu$ is defined similarly.

In addition to equation (1) for a hyperelastic material it is assumed the potential $\Psi(\mathrm{I}_\varepsilon, \mathrm{II}_\varepsilon, \mathrm{III}_\varepsilon)$ exists, such that [27]

$$\sigma = \nabla_\varepsilon \Psi(\mathrm{I}_\varepsilon, \mathrm{II}_\varepsilon, \mathrm{III}_\varepsilon). \tag{6}$$

Accounting relations (2), the condition (6) can be rewritten as [36]

$$\sigma = \frac{\partial \Psi}{\partial \mathrm{I}_\varepsilon}\boldsymbol{I} + \frac{\partial \Psi}{\partial \mathrm{II}_\varepsilon}\left(\mathrm{I}_\varepsilon \boldsymbol{I} - \varepsilon\right) + \frac{\partial \Psi}{\partial \mathrm{III}_\varepsilon}\left(\varepsilon\varepsilon - \mathrm{I}_\varepsilon \varepsilon + \mathrm{II}_\varepsilon \boldsymbol{I}\right). \tag{7}$$

Comparing equations (1) and (7) yields the following representation of Lame's constants in terms of the potential:

$$\lambda(\mathrm{I}_\varepsilon, \mathrm{II}_\varepsilon, \mathrm{III}_\varepsilon) = \frac{\partial \Psi}{\partial \mathrm{I}_\varepsilon}\mathrm{I}_\varepsilon^{-1} + \frac{\partial \Psi}{\partial \mathrm{II}_\varepsilon} + \frac{\partial \Psi}{\partial \mathrm{III}_\varepsilon}\mathrm{II}_\varepsilon \mathrm{I}_\varepsilon^{-1}, \qquad 2\mu(\mathrm{I}_\varepsilon, \mathrm{II}_\varepsilon, \mathrm{III}_\varepsilon) = -\frac{\partial \Psi}{\partial \mathrm{II}_\varepsilon} + \frac{\partial \Psi}{\partial \mathrm{III}_\varepsilon}\left(\varepsilon^{-1} - \mathrm{I}_\varepsilon\right). \tag{8}$$

Equations (8) impose some restrictions on behavior of the potential $\Psi$. In particular, since Lame's constants assumed to be continuous with respect to strain invariants, should be bounded at $\mathrm{I}_\varepsilon \to 0$ $\varepsilon \to 0$, equations (8) yields

$$\frac{\partial\Psi}{\partial \mathrm{I}_\varepsilon} = O(\mathrm{I}_\varepsilon),\ \mathrm{I}_\varepsilon \to 0; \qquad \frac{\partial\Psi}{\partial \mathrm{III}_\varepsilon} = O(\mathrm{I}_\varepsilon),\ \mathrm{I}_\varepsilon \to 0; \qquad \frac{\partial\Psi}{\partial \mathrm{III}_\varepsilon} = O(\mathrm{III}_\varepsilon),\ \mathrm{III}_\varepsilon \to 0. \tag{9}$$

At modeling of both statics and dynamics of granular materials, the hyperelastic constitutive equations are applied quite often [24]. It should be noted that in most of these works a concept of the multi-moduli media, actually, bi-modulus material, was applied [37] with a simple hyperelastic potential that is homogeneous of degree 2 with respect to the infinitesimal strain tensor

$$\Psi\left(\mathrm{I}_\varepsilon, \mathrm{II}_\varepsilon^\sim\right) \equiv \alpha \mathrm{I}_\varepsilon^2 + \beta \mathrm{II}_\varepsilon^\sim + \gamma \mathrm{I}_\varepsilon \sqrt{\mathrm{II}_\varepsilon^\sim}. \tag{10}$$

However, the discussed potential unfortunately, becomes irregular at vanishing second invariant. In the above equation, $\alpha, \beta, \gamma$ are the corresponding elastic material constants, independent of the invariants $\mathrm{I}_\varepsilon, \mathrm{II}_\varepsilon^\sim$

$$\mathrm{II}_\varepsilon^\sim = -\mathrm{II}_\varepsilon + \mathrm{I}_\varepsilon^2. \tag{11}$$

Introducing parameter $\gamma$ allows one to account dependence of material properties on sign of the first invariant.

It should also be noted that with introduction [23] of the potential

$$\Psi\left(\mathrm{I}_\varepsilon, \mathrm{II}_\varepsilon^\sim\right) = \Psi_1(\mathrm{I}_\varepsilon, \mathrm{II}_\varepsilon^\sim)(1 - \exp(-\chi(\mathrm{II}_\varepsilon^\sim));\ \chi(\mathrm{II}_\varepsilon^\sim) \to 0,\ \mathrm{II}_\varepsilon^\sim \to 0;\ \chi(\mathrm{II}_\varepsilon^\sim) \to \infty,\ \mathrm{II}_\varepsilon^\sim \to \infty, \tag{12}$$

where $\Psi_1(\mathrm{I}_\varepsilon, \mathrm{II}_\varepsilon^\sim)$ is an arbitrary potential, media with the dropdown (softening) diagrams can be modeled.

## 3. Elastic models

### *3.1. General equations*

Elastic models are described by the following equation of state

$$\boldsymbol{\sigma} = \lambda(\mathrm{I}_\sigma, \mathrm{II}_\sigma, \mathrm{III}_\sigma)\mathrm{I}_\varepsilon \boldsymbol{I} + 2\mu(\mathrm{I}_\sigma, \mathrm{II}_\sigma, \mathrm{III}_\sigma)\boldsymbol{\varepsilon}. \tag{13}$$

Compare this equation for the general nonlinear elastic and isotropic media at the infinitesimal deformations with equation (1) for the hyperelastic isotropic media.

### *3.2. Equations of motion*

By analogy with equation (4), the linearized equation of motion can be represented in a form

$$\frac{\lambda+2\mu}{\rho}\nabla_x \mathrm{div}_x u - \frac{\mu}{\rho}\mathrm{rot}_x\mathrm{rot}_x u + \underbrace{\frac{1}{\rho}\left[\nabla_x\lambda \mathrm{div}_x u + \nabla_x\mu(\nabla_x u + \nabla_x u^T)\right]}_{\text{lower order terms}} + b = \ddot{u}. \tag{14}$$

Despite the apparent more generality, the elastic models are rarely used for modeling granular materials. In [5, 17] problems related to the determination of velocities of acoustic waves in a granular media modeled by a system of elastic balls, interacting by the Hertz theory, were considered.

## 4. Hypoelastic models

### *4.1. General equations*

According to Trusedell [27] the time derivative of the stress tensor $\dot{\sigma}$ for a hypoelastic medium is determined by the time derivative of the strain tensor $\dot{\varepsilon}$. Assuming infinitesimal strains, the constitutive relation for an isotropic hypoelastic material can be written in a form

$$\dot{\sigma} = \lambda(\mathrm{I}_\sigma, \mathrm{II}_\sigma, \mathrm{III}_\sigma)\mathrm{I}_{\dot{\varepsilon}} \boldsymbol{I} + 2\mu(\mathrm{I}_\sigma, \mathrm{II}_\sigma, \mathrm{III}_\sigma)\dot{\varepsilon}. \tag{15}$$

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where $\dot{\sigma} = \frac{\partial \sigma}{\partial t}$; $\lambda$ and $\mu$ are functions of the corresponding invariants. Comparing the stress–strain relations for hypoelastic (15) and elastic media (13) reveals, the only difference is in the incremental form of the constitutive relation for the hypoelastic medium.

In theoretical works [26, 30] it was demonstrated that the special triggering mechanism can be incorporated into equation of state (15) allowing to account different states for active and unloading cases; thus, the general elastic-plastic behavior can be modeled within the hypoelastic models.

### *4.2. Equations of motion*

For a hypoelastic medium the equation of motion can be written in the form

$$\mathrm{div}\dot{\sigma} + \rho\dot{b} = \rho\ddot{v}, \tag{16}$$

where $\rho$ is the material density; it is assumed that $\dot{\rho} = 0$; $\dot{b}$ is the field of body forces. Substituting equation of state (15) into equation of motion (16) with account of the linearized Cauchy relations

$$\dot{\varepsilon} = \frac{1}{2}\left(\nabla_x v + \nabla_x v^T\right) \tag{17}$$

yields

$$\frac{\lambda+2\mu}{\rho}\nabla_x \mathrm{div}_x v - \frac{\mu}{\rho}\mathrm{rot}_x \mathrm{rot}_x v + \underbrace{\nabla_x \frac{\lambda}{\rho}\mathrm{div}_x v + \nabla_x \frac{\mu}{\rho}(\nabla_x v + \nabla_x v^T)}_{\text{lower order terms}} + \dot{b} = \ddot{v}, \tag{18}$$

where

$$\nabla_x \frac{\lambda}{\rho} = \frac{1}{\rho}\left(\frac{\partial\lambda}{\partial \mathrm{I}_\sigma}\nabla_x \mathrm{I}_\sigma + \frac{\partial\lambda}{\partial \mathrm{II}_\sigma}\nabla_x \mathrm{II}_\sigma + \frac{\partial\lambda}{\partial \mathrm{III}_\sigma}\nabla_x \mathrm{III}_\sigma\right). \tag{19}$$

The gradient $\nabla_x \mu$ is defined analogously.

Despite the obvious generality, the hypoelastic media are rarely used for modeling granular materials; in this regard it should be mentioned that the hypoelastic models were used for analyzing propagation of the impact bulk wave fronts propagating in granular materials [38], and the horizontally polarized surface acoustic waves; see [4].

## 5. Some inelastic models

### *5.1. General considerations*

Along with various elastic models, there is a large number of works accounting inelastic behavior of granular metamaterials. Apparently, one of the simplest inelastic models applicable for static and quasi static modeling of granular metamaterials are based on various variants of the Mohr–Coulomb and Drucker–Prager theories.

Remaining within a more traditional approach based on the Mohr–Coulomb plasticity model, several approaches can be mentioned that are used for applications in the mechanics of granular media; see for example [7]. The Mohr–Coulomb plasticity model is also applied to analyzing known effects of arising and developing inelastic strain prior to the extensive flow of the avalanches; see the experimental work [18].

### *5.2. Specific inelastic models for dynamics of granular metamaterials*

For the considered inelastic models used for dynamics of granular metamaterials apparently, the most widespread is the cam–clay (CC), the modified cam–clay (MCC) and the related models; see [1–3], along with some more recent works [6, 9]. For example, the ellipsoidal yield surface for the MCC model can be written as [9]

$$f(p, q_s, p_c) \equiv \frac{1}{\beta}\left(\frac{p}{a} - 1\right)^2 + \left(\frac{q_s}{Ma}\right)^2 - 1 = 0, \tag{20}$$

where $\beta$ is a dimensionless parameter specifying the ellipsoid shape: in a subcritical zone $\beta = 1$ (left side), in a supercritical zone $\beta \leqslant 1$ (right side); the dimensionless parameter $M$, known as the critical cone tangent, specifies ellipsoid dimension along $q_s$-axis; $a$ is the "central" point of the ellipsoid, this parameter defines ellipsoid dimension along $p$-axis:

$$a = \frac{p_c}{1+\beta}, \tag{21}$$

where $p_c$ is the current yield pressure value, note, that at $\beta = 1$ parameter $a$ takes value $p_c/2$. Actually, parameter $p_c$ specifies evolution of the ellipsoidal surface.

## 6. Concluding remarks

As the presented review shows, the hyperelastic equations of state are presumably, the most widespread for the use in characterization of the granular metamaterials behavior at the cyclic dynamic loadings.

Meanwhile, equation of state (10) for hyperelastic models is not the only equation of state used for characterization of the metamaterials having different moduli at the tension and compression phases. At the uniaxial motions some other potentials may be used, e.g., Morse and Lennard–Jones potentials; see [25].

The other problem of characterizing the analyzed granular metamaterials at acoustic wave propagation, associates with formation of the shock waves formation at the interfaces between tension and compression phases, where the bulk elastic moduli (and quite often shear moduli) become different.

One more problem is the structural heterogeneity, which is deliberately created to increase the energy dissipation by granular metamaterials. The corresponding phenomena are discussed in [10, 39, 40].

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